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Poisson algebras and transverse structures

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Abstract

Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra over an arbitrary field of characteristic zero. We study the transverse Poisson structure to a symplectic leaf of the dual space $\mathfrak{g}^*$, endowed with its canonical Poisson structure. We use two methods for this study, one is purely algebraic, the other is geometric. We prove that the transverse structure is a Poisson structure over the formal power series algebra in d indeterminates $k[X_1, \ldots, X_d]$, where d is the codimension of the symplectic leaf in $\mathfrak{g}^*$. We show a strong similarity between this Poisson structure and the associative algebra structure over this formal power series algebra introduced by Fokko du Cloux to describe the infinitesimal neighborhood of the corresponding representation via Kirillov's correspondence.

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0. Introduction

Let $\mathfrak{g}$ be a real finite-dimensional nilpotent Lie algebra. Let us consider a coadjoint orbit M in the dual space $\mathfrak{g}^*$. It is a symplectic leaf of the Poisson structure of $\mathfrak{g}^*$. The notion of transverse Poisson structure, which is a kind of Poisson algebra has been defined by Weinstein [36]. Moreover, Dixmier–Kirillov's theory [8] associates to M a primitive ideal J of the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Fokko du Cloux [11] has defined a notion of infinitesimal neighborhood of J the space Prim $U(\mathfrak{g})$ of the primitive ideals: it is an associative algebra. The underlying philosophy of this work is that Fokko du Cloux's algebra

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must be a quantization of the Poisson algebra defined by Weinstein. Let us consider the case $M = 0$. Then Weinstein's algebra is $S(\mathfrak{g})$, Fokko du Cloux's one $U(\mathfrak{g})$, which is rightfully considered as a quantization of $S(\mathfrak{g})$. In a general case, in the relation between Weinstein's algebra and that of Fokko du Cloux, one is less evident. However, we shall see from examples, there are convincing similarities.

Let us describe the two tackled problems. We fix a commutative field k with characteristic zero. Let $\mathfrak{g}$ be a nilpotent Lie algebra and M a coadjoint orbit in the dual space $\mathfrak{g}^*$ of the Lie algebra.

Let us consider $S(\mathfrak{g})$ the symmetric algebra of $\mathfrak{g}$ considered as the algebra of the polynomial functions on $\mathfrak{g}^*$. It is a Poisson algebra. Let I be the ideal formed by the functions vanishing on the orbit M. It is a Poisson ideal, $S(\mathfrak{g})/I$ is the algebra of the regular functions on M, provided with the Poisson bracket deduced from the Poisson structure of $S(\mathfrak{g})$ [35]. It has been proved by Vergne [35] for the generic case, by Arnal et al. [1] and Pedersen [25] in the general case, that $S(\mathfrak{g})/I$ is a Poisson–Weyl algebra: In other words, there exist $r \in \mathbb{N}$ and $2r$ elements $p_i, q_i, i \in \{1, \dots, r\}$ of $S(\mathfrak{g})/I$ such that

1. $S(\mathfrak{g})/I = k[p_1, \dots, p_r, q_1, \dots, q_r]$,
2. $\{p_i, p_j\} = \{q_i, q_j\} = 0, \quad \{p_i, q_j\} = \delta_{i,j}$ (Darboux's relations).

The first problem which we considered is the following one. The functions p_i, q_i are the restrictions of polynomial functions f_i, g_i defined on $\mathfrak{g}^*$. It is natural to wonder whether we can choose extensions such that Darboux's relations are still being satisfied. In algebraic terms, does there exist a homomorphism of Poisson algebras $S(\mathfrak{g})/I \longrightarrow S(\mathfrak{g})$ which is a right inverse to the canonical projection $S(\mathfrak{g}) \longrightarrow S(\mathfrak{g})/I$?

The simplest examples show that it is not true. However, we can often find rational functions f_i, g_i defined on an invariant Zariski open set of $\mathfrak{g}^*$ including M and still satisfying Darboux's relations: for example such is the case if the orbit M is generic [35] or, at the extreme opposite if $\dim M = 0$.

Although we cannot formally prove it, in general, it appears impossible. Here, we prove that we can find "functions" $\widehat{f_i}$, $\widehat{g_i}$, defined in a formal neighborhood of M, still satisfying Darboux's relations. In other words, the projective limit $\widehat{S}(\mathfrak{g}) = \varprojlim_n S(\mathfrak{g})/I^n$ has a natural structure of Poisson algebra, and the $\widehat{f_i}$, $\widehat{g_i}$ are in $\widehat{S}(\mathfrak{g})$.

This theorem is inspired and motivated by similar results of Fokko du Cloux on the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Let us consider a rational ideal J of $U(\mathfrak{g})$ (this is a prime ideal such that the center of the fraction field of $U(\mathfrak{g})/J$ is equal to k [8]). Dixmier–Kirillov's theory asserts that $U(\mathfrak{g})/J$ is a Weyl algebra, generated by $2r$ generators a_i, b_i, $i \in \{1, \dots, r\}$ (r is the weight of J [8]) satisfying commutation relations

$$[a_i, a_j] = [b_i, b_j] = 0, \qquad [a_i, b_j] = \delta_{i,j}, \quad i, j \in \{1, \dots, r\}.$$

Fokko du Cloux proved that in the completion $\widehat{U}(\mathfrak{g}) = \varprojlim_n U(\mathfrak{g})/J^n$ we can find elements $\widehat{a_i}$, $\widehat{b_i}$ projecting itself by canonical projection $\widehat{U}(\mathfrak{g}) \longrightarrow U(\mathfrak{g})/J$ on the previous elements, and still satisfying the same commutation relations.

Our result is related to those of Weinstein's. Let us assume that $k = \mathbb{R}$. Let us fix an element μ of M. Then there exists an open neighborhood of μ in $\mathfrak{g}^*$ (for the usual topology), and in V analytic functions f_i, g_i extending the p_i, q_i, still satisfying Darboux's relations. The comparison of these two results, and examples that we have calculated, suggests there exists a neighborhood V of M in $\mathfrak{g}^*$ (for the usual topology), and in this open analytic functions (and may be algebraic) f_i^M, g_i^M extending the functions p_i, q_i, still satisfying Darboux's relations and such that, moreover, the restriction of partial derivatives to M are polynomial.

The second problem which is studied concerns the transverse structures. Let us denote C the commutant of the lifting of the f_i, and g_i:

$$C = \{\hat{c} \in \widehat{S}(\mathfrak{g}) \mid \{\hat{c}, \hat{f}_i\} = 0, \{\hat{c}, \hat{g}_i\} = 0 \quad \forall i \in \{1, \ldots, r\}\}.$$

Let h be the codimension of M in $\mathfrak{g}^*$. We prove that C is an algebra of formal series in h variables, provided with a structure of Poisson algebra singular at 0. Its isomorphism class does not depend on the considered choices. We prove that for ($k = \mathbb{R}$), this algebra is the algebra obtained from Weinstein's one (it is a Poisson structure in a neighborhood of zero in $\mathbb{R}^h$, singular at 0) if we consider Taylor's series at the origin. The use of Dirac's brackets formula allows the calculation of the commutant C with an extreme simplicity. The transverse structure is given, in convenient coordinates, by rational functions. We show that the graded algebra of C in its maximal element is isomorphic to the symmetric algebra of the stabilizer $\mathfrak{g}_\mu$ at a point of the considered orbit. Our construction of C is similar to Fokko du Cloux's construction. He considers the commutant D of the elements $\hat{u}_i$, $\hat{v}_i$ in the completion $\hat{U}(\mathfrak{g})$:

$$D = \{\hat{d} \in \hat{U}(\mathfrak{g}) \mid [\hat{d}, \hat{u}_i] = 0, [\hat{d}, \hat{v}_i] = 0 \quad \forall i \in \{1, \ldots, r\}\}.$$

In general this commutant is a non-commutative algebra of formal series in h variables. This formal similarity between C and D, as well as similarities between the formulae (see examples), are a first justification to our assertion "Fokko du Cloux's algebra is a quantization of Alan Weinstein's transverse structure".

1. Definitions and some properties of Poisson algebras and Poisson modules

1.1. Poisson algebras and Poisson ideals

Let us recall that a Poisson algebra is an associative unitary commutative algebra A endowed with a bilinear map $A \times A \longrightarrow A$ denoted by $\{\cdot, \cdot\}$, called Poisson brackets, providing A with a Lie algebra structure and satisfying the relation $\{ab, c\} = a\{b, c\} + \{a, c\}b$ for all a, b, c in A.

In A, two elements a and b are said to commute if $\{a, b\} = 0$. A Poisson ideal of a Poisson algebra A is an ideal of A considered as an associative algebra and an ideal of A considered as a Lie algebra.

Example 1. Let $\mathfrak{g}$ be a Lie algebra, the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$ has a structure of Poisson algebra defined by the bracket $\{x, y\} = [x, y]$ for all x, y of $\mathfrak{g}$. Let $\mathcal{A}$ the adjoint algebraic group $\mathfrak{g}$. Any $\mathcal{A}$-invariant ideal of the symmetric algebra $S(\mathfrak{g})$ is a Poisson ideal.

For all subset X of the dual $\mathfrak{g}^*$ of $\mathfrak{g}$, $\mathcal{A}$-invariant by the contragradient action, the ideal $I(X)$ of the zeros of X is a Poisson ideal. The algebra $S(\mathfrak{g})/I(X)$ is a Poisson algebra.

When I is a Poisson ideal of A, the bracket $\{I, I\}$ is a Lie ideal but is not generally a Poisson ideal: the condition $A\{I, I\} \subset \{I, I\}$ may not be satisfied. (Examples are given in [31].)

In part 2, we shall use the descending series of a Poisson ideal I defined by $\mathfrak{D}_1 I = I$ and $\mathfrak{D}_{n+1} I = \{I, \mathfrak{D}_n\}$ for $n \geq 1$. They are Lie ideals. We shall use the Lie ideals $\{I^n, I^m\}$, $n, m \in \mathbb{N}$. Provided with the Poisson bracket, I/I^2 is a Lie algebra over the ring A/I.

Proposition 2. *We have*
(1) $\{I^n, I^m\} \subset I^{n+m-1}$, *for n and $m \in \mathbb{N}$* $(I^{-1} = I^0 = A)$
(2) (a) $\mathfrak{D}_n I + I^2$ *is a Poisson ideal, for* $n \geq 1$,
(b) $\{\mathfrak{D}_n I + I^2, \mathfrak{D}_m I + I^2\} \subset \mathfrak{D}_{n+m} I + I^2$.

1.2. Poisson modules, Poisson cohomology, Poisson extension of a Poisson algebra

1.2.1. Definition of a Poisson module over a Poisson algebra

Definition 3. Let A be a Poisson algebra. Let M a vector space and End M be the algebra of the endomorphisms of M. The vector space M is called a Poisson A-module if we have:
(1) A homomorphism of associative unitary algebras $\rho : A \longrightarrow \operatorname{End} M$; we shall denote $\rho(a)(m) = a \cdot m$, $a \in A$, $m \in M$.
(2) A homomorphism of Lie algebras $\omega : A \longrightarrow \operatorname{End} M$; we shall denote $\omega(a)(m) = [a, m]$, $a \in A$, $m \in M$.
(3) The following compatibility relations between the structures defined by ρ and ω:

$$[b, a \cdot m] = a \cdot [b, m] + \{b, a\} \cdot m, \quad [ab, m] = a \cdot [b, m] + b \cdot [a, m].$$

Example 4. A is a Poisson A-module. More generally, if I is a Poisson ideal, I and A/I are Poisson A-modules.

Property 5. *Let M and N be Poisson A-modules. Then the tensor product $M \otimes_A N$ is a Poisson A-module provided with the structure:*
1. $a, m \otimes_A n = am \otimes_A n = m \otimes_A an$;
2. $[a, m \otimes_A n] = [a, m] \otimes_A n + m \otimes_A [a, n], \quad a \in A, m \in M, n \in N.$

We have a correspondence between Poisson module and Rinehart module. For further information, cf. [16,24,30,31].

1.2.2. Definition of a cochain complex and of the cohomology $H^\star_{\mathrm{Poisson}}(A, M)$: comparison of $H^\star_{\mathrm{Poisson}}(A)$ with $H^\star_{\mathrm{de\,Rham}}(A)$

The cohomology of a Poisson algebra A has been defined by Lichnerowicz [19] when the algebra A is the algebra of functions $C^\infty(N)$, N being a Poisson differentiable variety C^∞. For the general case, see [16,33].

Property 6. *Let M be a Poisson A-module and $n \in \mathbb{N}$. Let us denote $\mathrm{Alt}^{n,D}(A, M)$ the vector space defined by for $n = 0$, $\mathrm{Alt}^{0,D}(A, M) = M$, for $n \geq 1$, $\mathrm{Alt}^{n,D}(A, M) = \{f : A^n \longmapsto M;\ f$ is multilinear alternating, and a derivation of A into M in each variable$\}$. Let $\mathrm{Alt}^D(A, M) = \oplus_{n\geq 0} \mathrm{Alt}^{n,D}(A, M)$. Let $d : \mathrm{Alt}^D(A, M) \longrightarrow \mathrm{Alt}^D(A, M)$ be the endomorphism defined by*

$$d_0 v(a_1) = [a_1, v] \text{ if } v \in \mathrm{Alt}^{0,D}(A, M) = M,$$

$$d_n v(a_1, a_2, \ldots, a_{n+1}) = \sum_{i=1}^{i=n+1} (-1)^{i+1} [a_i, v(a_1, \ldots, \widehat{a_i}, \ldots, a_{n+1})] + \sum_{1\leq i<j\leq n+1} (-1)^{i+j} v(\{a_i, a_j\}, a_1, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, a_{n+1}), \quad \text{if } n > 0,\ \forall a_i \in A.$$

Then the couple (Alt^D, d) is a complex of vector spaces of degree $+1$.

We denote by $H^\star_{\text{Poisson}}(A, M)$ the cohomology of the the complex (Alt^D, d), called cohomology of the Poisson algebra with values in M. For $M = A$, $H^\star_{\text{Poisson}}(A, A)$ will be called the cohomology of the Poisson algebra A and will be denoted $H^\star_{\text{Poisson}}(A)$.

1.2.3. Comparison of the G. de Rham complex of a Poisson algebra with its Poisson complex

Let S be an associative unitary and commutative algebra, Ω_S the S-module of the differentials of S over k and $d : S \longrightarrow \Omega_S$ the associated derivation [3, Chap. 3, p. 134]. Let $\Lambda(\Omega_S)$ the exterior algebra graded with the pth exterior powers $\Lambda^p(\Omega_S)$ of the S-module Ω_S and $(d^\bullet, \Lambda(\Omega_S))$ the de Rham complex of S over k. We denote $H^\star_{\text{de Rham}}(S)$ the associated cohomology [3, Chap. 10, p. 43].

In [31] we describe a morphism of complexes ϕ from the de Rham complex of A into the Poisson complex $(d, \mathrm{Alt}^D(A, A))$. We deduce by passage to the quotient the homomorphism of graded vector spaces $H(\phi) : H^\star_{\text{de Rham}}(A) \longrightarrow H^\star_{\text{Poisson}}(A)$. There is an important case where $H(\phi)$ is an isomorphism: the algebra A is a symplectic algebra over k. Let S be an associative unitary commutative algebra and an element X in $\mathrm{Der}\, S$. There exists one and only one odd derivation $i_X : \Lambda(\Omega_S) \longrightarrow \Lambda(\Omega_S)$ of degree -1 such that $i_X(\mathrm{d}s) = X(s)$, for all s in S.

Let $\omega \in \Lambda^2(\Omega_S)$. The couple (S, ω) is called a symplectic algebra (see [20, Definition 1.3]) if:

1. $\mathrm{d}\omega = 0$,
2. the map $I_\omega : \mathrm{Der}\, S \longrightarrow \Omega_S$ defined by $I_\omega(X) = i_X\omega$ for $X \in \mathrm{Der}\, S$ is an isomorphism of S-modules.

We shall denote X_b the unique derivation such that $i_{X_b}\omega = \mathrm{d}b$. Any symplectic algebra (S, ω) is a Poisson algebra provided with the bracket $\{a, b\} = i_{X_a}(i_{X_b}\omega)$ [20]. We have $\{a, b\} = i_{X_a}(\mathrm{d}b)$ by definition of X_b and hence $\{a, b\} = X_a(b)$. We have the following property stated by Lichnerowicz [19, p. 259] in the context of the Poisson manifolds which are symplectic:

Property 7. *Let (S, ω) be a symplectic algebra, then the de Rham and Poisson cohomologies S, $H^{\star}_{\text{de Rham}}(S)$ et $H^{\star}_{\text{Poisson}}(S)$, are isomorphic.*

Proof. We show that $H(\phi)$ is an isomorphism of complexes (see [31, p. 16]). □

1.2.4. Extensions of Poisson algebra

Definition 8. Let B be a Poisson algebra and M a Poisson B-module. A Poisson extension of B by M is the data of a Poisson algebra A and of an exact sequence of vectors spaces (ξ)

$$(\xi) : 0 \longrightarrow M \xrightarrow{g} A \xrightarrow{f} B \longrightarrow 0$$

such that the map f is a Poisson algebra homomorphism and where the homomorphism of vector spaces g satisfies

$$(\star) \quad g(b.m) = ag(m) \quad \text{et} \quad g([b, m]) = \{a, g(m)\},$$

with $m \in M, b \in B$ and $a \in A$ such that $f(a) = b$.

The extension (ξ) of B by M will be called inessential if there exists a homomorphism of Poisson algebras $u : B \longrightarrow A$ such that $f \circ u = \mathrm{Id}_B$.

The Poisson extension (ξ) of B by M will be called a split extension by a homomorphism of associative algebras if there exists $h : B \longrightarrow A$ homomorphism of associative algebras satisfying $f \circ h = \mathrm{Id}_B$.

The set of equivalence classes of split extensions of B by M will be denoted by $EX(B, M)$.

The set $EX(B, M)$ is in bijection with $H^2_{\text{Poisson}}(B, M)$. We have the following theorem:

Theorem 9.

(1) *If a belongs to $Z^2(B, M)$, then the Poisson extension of B by M:*

$$(\xi) : 0 \longrightarrow M \xrightarrow{i} (B \oplus M)_a \xrightarrow{p} B \longrightarrow 0$$

associated to a where $(B \oplus M)_a$ is the vector space $B \oplus M$ endowed with the structure of Poisson algebra defined by

$$(\star) \quad (b, m)(b', m') = (bb', bm' + b'm) \quad et \quad \{(b, m)(b', m')\}$$
$$= (\{b, b'\}, [b, m'] - [b', m] + a(b, b')),$$

$b, b' \in B$, $m, m' \in M$ and $i(m) = (0, m)$ $p(b, m) = b$, belongs to $EX(B, M)$.

(2) *If $(\xi) : 0 \longrightarrow M \xrightarrow{g} A \underset{s}{\overset{f}{\rightleftarrows}} B \longrightarrow 0$ is a split Poisson extension of B by M in $EX(B, M)$ with the split homomorphism of associative algebras $s : B \longrightarrow A$, then we can associate to (ξ) a 2-cocycle, ω, element of $Z^2_{\text{Poisson}}(B, M)$ defined by*

$$g(\omega(x, y)) = \{s(x), s(y)\} - s(\{x, y\}), \quad x, y \in B.$$

Moreover, if s' is another homomorphism of associative algebras satisfying the equality $f \circ s' = \mathrm{Id}_B$, the cocycle ω' associated to s' satisfies $\omega - \omega' = d_1 u$, u being the derivation of B into M defined by $g \circ u = s - s'$. We have in $H^2_{\mathrm{Poisson}}(B, M)$, the equality $\overline{\omega} = \overline{\omega}'$.

(3) *We have a bijection of $H^2_{\mathrm{Poisson}}(B, M)$ onto $EX(B, M)$; the inessential extensions form a unique class in $EX(B, M)$ corresponding to the zero in $H^2_{\mathrm{Poisson}}(B, M)$.*

Proof. (1) and (2) are easily proved.

(3) Let $\Phi : H^2_{\mathrm{Poisson}}(B, M) \longrightarrow EX(B, M)$ the map such that $\Phi(\overline{a}) = (\overline{\xi})$, where $\overline{\xi}$ is defined by

$$(\xi) : 0 \longrightarrow M \xrightarrow{i} (B \oplus M)_{(a)} \xrightarrow{p} B \longrightarrow 0.$$

Let $\varphi : EX(B, M) \longrightarrow H^2_{\mathrm{Poisson}}(B, M)$ the map which associates to $(\overline{\xi}) : 0 \longrightarrow M \xrightarrow{g} A \underset{s}{\overset{f}{\rightleftarrows}} B \longrightarrow 0$ the equivalence class $\overline{\omega}$ where ω is defined by $g(\omega(x, y)) = \{s(x), s(y)\} - s(\{x, y\})$, $s : B \longrightarrow A$ being the homomorphism of associative algebras satisfying the equality $f \circ s = \mathrm{Id}_B$. We verify that Φ and φ are well defined (see [31]) and that we have $\varphi \circ \Phi = \mathrm{Id}_{EX(B,M)}$ and $\Phi \circ \varphi = \mathrm{Id}_{H^2(B,M)}$. □

1.3. *The Weyl algebra $W_n(k)$*

1.3.1. *Definition of the Poisson–Weyl algebra $W_n(k)$ and of its cohomology*

A commutative algebra generated by the family $(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$ provided with the structure of Poisson algebra such that

$$\forall i, j \in \{1, \ldots, n\}, \quad \{X_i, Y_j\} = \delta_{ij}, \quad \{X_i, X_j\} = \{Y_i, Y_j\} = 0$$

is called a Poisson–Weyl algebra of order n, and denoted by $W_n(k)$.

The Poisson–Weyl algebra $W_n(k)$ is thus the algebra $k[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ provided with the bracket

$$\{f, g\} = \sum_{i=1}^{i=n} \left(\frac{\partial f}{\partial X_i} \frac{\partial g}{\partial Y_i} - \frac{\partial f}{\partial Y_i} \frac{\partial g}{\partial X_i} \right) \forall f, \quad g \in k[X, Y].$$

A Poisson algebra is said to be simple if it has no other Poisson ideal than (0) and A. We show that the Poisson–Weyl algebra $W_n(k)$ is simple and that its center is the field k. The Poisson–Weyl algebra is a symplectic algebra $(W_n(k), \omega = \sum_{i=1}^{i=n} \mathrm{d}X_i \wedge \mathrm{d}Y_i)$ which the associated Poisson bracket is the bracket of $W_n(k)$ as Poisson algebra.

Proposition 10. *Let $W_n(k)$ be the Poisson–Weyl algebra of order n. Then*

(1) $H^0_{\mathrm{Poisson}}(W_n(k), W_n(k)) = k$,

(2) *for all* $p \geq 1$ $H^p_{\mathrm{Poisson}}(W_n(k), W_n(k)) = 0$.

Proof. The Poisson and de Rham cohomologies coincide by virtue of Property 7. The de Rham cohomology of $W_n(k)$ is acyclic in degree greater than zero from Nicolas Bourbaki's theorem [3, Chap. 10, p. 159 example]. □

1.4. Centralizing sequence of a Poisson A-module M: properties of nilpotence

Let A be a Poisson algebra, M a Poisson A-module. By definition

$$H^0(A, M) = \{m \in M: \forall a \in A \quad [a, m] = 0\}.$$

Each element of $H^0(A, M)$ will be said to be central. Let $(x_1, \ldots, x_n)$ a sequence of elements of M. We say that the sequence of elements $(x_1, \ldots, x_n)$ of M is centralizing if x_1 is central and if we have the inclusion, for all $i \geq 2$,

$$[A, x_i] \subset Ax_{i-1} + \cdots + Ax_1.$$

In this case, for all $h \in \{1, \ldots, n\}$, $I_h = Ax_h + \cdots + Ax_1$ is a Poisson A-submodule of M. We check this easily (cf. [31]).

Proposition 11. *Let $(x_1, \ldots, x_n)$ be a centralizing sequence. Let us set $I_h = Ax_h + \cdots + Ax_1$, $h \in \{1, \ldots, n\}$ et $I_0 = \{0\}$. Then we have*
(1) $\{I_n, I_h\} \subset I_n I_h + I_{h-1}$, $\forall h \geq 0 (I_{-1} = 0)$.
(2) $\forall p \geq 2$, $\mathfrak{D}_p I_n \subset I_n^2 + I_{n-p+1}$.

Corollary 12. *Let I be a Poisson ideal. We suppose that I is generated by a centralizing sequence. Then:*
(1) *if the length of the centralizing sequence is h, then*

$$\mathfrak{D}_{h+1} I \subset I^2;$$

(2) *for all $n \in \mathbb{N}$, there exists $p \in \mathbb{N}$ such that $\mathfrak{D}_p I \subset I^n$.*

Let $\mathfrak{g}$ be a Lie algebra of a linear algebraic group and $\mathcal{A}$ its adjoint group. Let M be a coadjoint orbit of $\mathcal{A}$ in $\mathfrak{g}^*$ and I the Poisson ideal of the Poisson algebra $S(\mathfrak{g})$ associated to this orbit. We know that the $S(\mathfrak{g})/I$-module I/I^2 is a Lie $S(\mathfrak{g})/I$-algebra. Let $P(M, k)$ be the algebra of polynomial functions on the orbit. The algebra $P(M, \mathfrak{g})$ of the polynomial functions on M with values in $\mathfrak{g}$ is a Lie algebra over the ring $P(M, k)$, The structure of $P(M, k)$-module is defined naturally, the multiplication by

$$[f, h](\nu) = [f(\nu), h(\nu)], \quad f, h \in P(M, \mathfrak{g}), \quad \nu \in M.$$

The Lie algebra $P(M, \mathfrak{g})$ is isomorphic to the Lie $S(\mathfrak{g})/I$-algebra $S(\mathfrak{g})/I \otimes \mathfrak{g}$ obtained by extension. We show that the Lie algebra I/I^2 is a Lie subalgebra of $S(\mathfrak{g})/I \otimes \mathfrak{g}$.

Proposition 13. *Let M be a coadjoint orbit of $\mathfrak{g}^*$ and I the associated ideal. Let $P(M, \mathfrak{g}) \simeq S(\mathfrak{g})/I \otimes \mathfrak{g}$ the Lie algebra over $S(\mathfrak{g})/I$ of the polynomial functions on the orbit M with values in $\mathfrak{g}$.*

Then the map $\phi : I/I^2 \longrightarrow S(\mathfrak{g})/I \otimes \mathfrak{g}$ such that for f in I and ν in M

$$\phi(f + I^2)(\nu) = d_\nu f$$

is an injective homomorphism of Lie $S(\mathfrak{g})/I$-algebra.

Proof. Let $\psi : I \longrightarrow P(M, \mathfrak{g})$ be the function such that we have, for h in I and ν in M, $\psi(h)(\nu) = d_\nu h$. For all ν in M, the tangent space $T_\nu M$ of M at ν is the orthogonal $\mathfrak{g}_\nu^\perp$ of the stabilizer $\mathfrak{g}_\nu$ of ν in $\mathfrak{g}^*$. We deduce, h vanishing on M, that the linear map $d_\nu h$ vanishes on $\mathfrak{g}_\nu^\perp$, that is that we have $d_\nu h \in (\mathfrak{g}_\nu^\perp)^\perp = \mathfrak{g}_\nu$.

Let us verify that the map ψ is defined by passage to the quotient. We have $\psi(I^2) = 0$. Indeed for f, g in I and ν in M it appears that $\psi(fg)(\nu) = d_\nu(fg) = f(\nu)d_\nu g + g(\nu)d_\nu f = 0$. Thus the map $\phi : I/I^2 \longrightarrow S(\mathfrak{g})/I \otimes \mathfrak{g}$ is well defined, we verify that it is $S(\mathfrak{g})/I$-linear. It is injective, since we have $f(\nu) = d_\nu f = 0$ for $\nu \in M$. Let us verify that for all f et g in I we have $\phi(\{f + I^2, g + I^2\}) = [\phi(f + I^2), \phi(g + I^2)]$, that is that $\mathrm{d}\{f, g\}(\nu) = [d_\nu f, d_\nu g]$, $\nu \in M$ (for a particular case of this formula see [13, p. 585, 2°]). By definition of the bracket in $S(\mathfrak{g})$, we have

$$\{f, g\}(\nu) = \nu([d_\nu f, d_\nu g]), \qquad \nu \in M,$$

and by definition of the derivative it appears that

$$\mathrm{d}_\nu\{f, g\}.h = \frac{\{f, g\}(\nu + th) - \{f, g\}(\nu)}{t} \quad (\mathrm{mod}\, t), \quad h \in \mathfrak{g}^*.$$

We must evaluate

$$\{f, g\}(\nu + th) - \{f, g\}(\nu) = (\nu + th)([d_{\nu+th} f, d_{\nu+th} g] - \nu([d_\nu f, d_\nu g]). \tag{$\star$}$$

We have

$$\begin{aligned} d_{\nu+th} f - d_\nu f &= t d_\nu^2 f.h \quad (\mathrm{mod}\ t^2), \\ d_{\nu+th} g - d_\nu f &= t d_\nu^2 g.h \quad (\mathrm{mod}\ t^2). \end{aligned}$$

Reporting in ($\star$), we use the linearity of the bracket in $\mathfrak{g}$ and the fact that $\mathrm{d}f(\nu)$ and $\mathrm{d}g(\nu)$ belong to $\mathfrak{g}_\nu$, we obtain $\{f, g\}(\nu + th) - \{f, g\}(\nu)/t = h([d_\nu f, d_\nu g])$ (mod t). □

Corollary 14. *Let $\mu \in \mathfrak{g}^*$. Let us assume that the stabilizer of μ is nilpotent: $\mathfrak{D}_n \mathfrak{g}_\mu = 0$ for some n of $\mathbb{N}$. If I is the associated ideal of the orbit through μ, then we have $\mathfrak{D}_n I \subset I^2$.*

Proof. Indeed for all f in I/I^2, its image by ϕ at a point ν of M belongs to $\mathfrak{g}_\nu$. □

2. Lifting map, commutant and graded algebra

2.1. General lifting Poisson homomorphism

2.1.1. A lifting theorem

Theorem 15. *Let B be a Poisson algebra and M a Poisson B-module. Let (ξ) an extension of B by M $(\xi) : 0 \longrightarrow M \xrightarrow{g} A \xrightarrow{f} B \longrightarrow 0$. Let C be a Poisson algebra and $h : C \longrightarrow B$ a homomorphism of Poisson algebras. Let us assume that there exists a homomorphism of associative algebras $s : C \longrightarrow A$ such that we have $f \circ s = h$ and $H^2_{\text{Poisson}}(C, M) = 0$.*

Then

(1) *there exists $r : C \longrightarrow A$ a homomorphism of Poisson algebras such that $f \circ r = h$*

$$\begin{tikzcd}(\xi) : 0 \arrow[r] & M \arrow[r, "g"] & A \arrow[r, "f"] & B \arrow[r] & 0 \\ & & & C \arrow[ul, "r, s"] \arrow[u, "h"'] & \end{tikzcd}$$

(2) *if moreover $H^1_{\text{Poisson}}(C, M) = 0$, then two homomorphisms of Poisson algebras, r and r' such that $f \circ r = h$ et $f \circ r' = h$, satisfy the property:*
there exists m in M such that for all x of C

$$r'(x) = r(x) + \{r(x), g(m)\}.$$

Proof. See [31, p. 29]. □

2.1.2. Study of an isomorphism of Poisson modules

Let M be a Poisson A-module. We endow the A-module $A \otimes H^0(A, M)$ of the structure of Poisson A-module defined by: $[a, b \otimes m] = \{a, b\} \otimes m$.

The following theorem is similar to Fokko du Cloux's result [11, Lemma 4.2.1, p. 178].

Theorem 16. *Let us assume that M is generated by a centralizing sequence $(x_1, \dots, x_n)$ and that A is isomorphic to a Poisson–Weyl algebra.*

Then the linear map $\phi : A \otimes H^0_{\text{Poisson}}(A, M) \longrightarrow M$ such that $\phi(a \otimes m) = a \cdot m$ with $a \in A$ and $m \in M$ is an isomorphism of Poisson A-modules. Moreover, the dimension of $H^0_{\text{Poisson}}(A, M)$ is finite.

Proof. We shall show by induction on n that ϕ is a bijection. For $n = 1$, the surjection of ϕ is obvious and the injection of ϕ_1 is obtained by the simplicity of A. The induction hypothesis states thus: Let us assume that for the A-module M' generated by a centralizing sequence having for length $n - 1$, we have an isomorphism of Poisson A-module $A \otimes_k H^0(A, M') \simeq M'$. Then let $M = Ax_1 + \cdots + Ax_n$ be a module generated by a centralizing sequence having a length n.

Since Ax_1 is a submodule, M/Ax_1 is a Poisson A-module generated by a centralizing sequence $(\overline{x_2}, \overline{x_3}, \dots, \overline{x_n})$ with $\overline{x_i} = x_i + Ax_1$, $i \in \{2, \dots, n\}$ of length $n - 1$. From the short exact sequence of Poisson A-module

$$0 \longrightarrow Ax_1 \xrightarrow{i} M \xrightarrow{p} M/Ax_1 \longrightarrow 0,$$

we deduce the long exact sequence

$$0 \longrightarrow H^0_{\text{Poisson}}(A, Ax_1) \longrightarrow H^0_{\text{Poisson}}(A, M) \longrightarrow H^0_{\text{Poisson}}(A, M/Ax_1) \longrightarrow H^1_{\text{Poisson}}(A, Ax_1) \longrightarrow \cdots$$

If we have $x_1 = 0$, then it appears that the equality $H^1_{\text{Poisson}}(A, Ax_1) = 0$. Otherwise the isomorphism $Ax_1 \simeq A$ and the fact that $H^1_{\text{Poisson}}(A, Ax_1) = H^1_{\text{Poisson}}(A, A)$ vanishes, is a consequence of Proposition 10.

We have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \otimes H^0_{\text{Poisson}}(A, Ax_1) \arrow[r] \arrow[d, equal] & A \otimes H^0_{\text{Poisson}}(A, M) \arrow[r] \arrow[d, "\phi"] & A \otimes H^0_{\text{Poisson}}(A, M/Ax_1) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Ax_1 \arrow[r] & M \arrow[r] & M/Ax_1 \arrow[r] & 0
\end{tikzcd}$$

From it we infer that ϕ is an isomorphism.

Let us show that dim $H^0_{\text{Poisson}}(A, M)$ est finite. When $M = Ax_1$, then $H^0_{\text{Poisson}}(A, M) = kx_1$. An induction similar to the previous one achieves the proof. □

2.2. *Lifting of A/I*

In the rest of the paper, for a Poisson algebra A and I a Poisson ideal of A such that every Poisson ideal of A is generated by a centralizing sequence and such that the algebra A/I is isomorphic to Poisson–Weyl algebra, we shall say that "*A and I satisfy the lifting hypothesis*". Let $\hat{A} = \varprojlim_n A/I^n$ the projective limit, it is a Poisson algebra. We shall denote $f_n : \hat{A} \longrightarrow A/I^n$ the canonical projection.

2.2.1. *Construction of a lifting homomorphism from A/I into $\hat{A}$*

Theorem 17 (Lifting theorem). *Let A and I satisfy the lifting hypothesis. Then there exists a homomorphism of Poisson algebras $\hat{\varphi} : A/I \longrightarrow \hat{A}$ called lifting homomorphism such that $f_1 \circ \hat{\varphi} = \mathrm{Id}_{A/I}$.*

Proof. We proceed in two steps. □

First step. We shall lift A/I into A/I^2. For that, we use the decreasing sequence of Poisson ideals $I^2 + \mathfrak{D}_n I$, $n \geq 1$, from Proposition 2. In the first instance, we have the exact sequence of Poisson algebras

$$(\xi)\quad \begin{tikzcd} 0 \arrow[r] & I/(\mathfrak{D}_2 I + I^2) \arrow[r, "i_2"] & A/(\mathfrak{D}_2 I + I^2) \arrow[r, "\varphi_{1,2}"] & A/I \arrow[r] & 0 \\ & & & A/I \arrow[u] & \end{tikzcd}$$

We apply Theorem 15 to this diagram. It is clear that (ξ) is a Poisson extension of A/I by $I/(\mathfrak{D}_2 I + I^2)$. Thus $I/(\mathfrak{D}_2 I + I^2)$ is a Poisson A/I-module. The A/I is a Poisson–Weyl algebra, hence it is a commutative algebra generated by an algebraic independent family over k; by an arbitrary lifting of the generators of this algebra, there exists a homomorphism of associative algebras such that $\varphi_{12} \circ s = \mathrm{Id}_{A/I}$. It remains to check $H^2_{\text{Poisson}}(A/I, I/(\mathfrak{D}_2 I + I^2)) = 0$. By hypothesis, the Poisson A/I module $I/(\mathfrak{D}_2 I + I^2)$ is generated by a centralizing sequence. From Theorem 16 applied to $M = I/(\mathfrak{D}_2 I + I^2)$, we obtain the isomorphism of Poisson A/I-module $A/I \otimes H^0_{\text{Poisson}}(A/I, I/(\mathfrak{D}_2 I + I^2)) \simeq I/(\mathfrak{D}_2 I + I^2)$. The module $I/(\mathfrak{D}_2 I + I^2)$ is a direct sum of dim $H^0_{\text{Poisson}}(A/I, I/(\mathfrak{D}_2 I + I^2))$ copies of A/I. On the other hand $H^2_{\text{Poisson}}(A/I, A/I) = 0$ from Proposition 10. Hence we have $H^2_{\text{Poisson}}(A/I, I/(\mathfrak{D}_2 I + I^2)) = 0$. Theorem 15 affirms the existence of a homomorphism of Poisson algebras $r_2 : A/I \to A/(\mathfrak{D}_2 I + I^2)$ such that $\varphi_{1,2} \circ r_2 = \mathrm{Id}_{A/I}$.

Once again, we apply the same theorem to the diagram for $p \geq 3$

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{D}_p I + I^2/\mathfrak{D}_{p+1} I + I^2 \arrow[r] & A/\mathfrak{D}_{p+1} I + I^2 \arrow[r, "\varphi_{p,p+1}"] & A/\mathfrak{D}_p I + I^2 \arrow[r] & 0 \\ & & & A/I \arrow[u, "r_p"'] \arrow[ul, "r_{p+1}"] & \end{tikzcd}$$

and we obtain r_{p+1} homomorphism of Poisson algebras such that $\varphi_{p,p+1} \circ r_{p+1} = r_p$. If the natural number n is the length of the centralizing sequence generating I, we use the property of nilpotence

$$\mathfrak{D}_{n+1} I + I^2 = I^2$$

from Corollary 12. We have $\varphi_{1,2} \circ r_2 = \mathrm{Id}_{A/I}$, $\varphi_{2,3} \circ r_3 = r_2, \ldots, \varphi_{n,n+1} \circ r_{n+1} = r_n$. And thus we obtain $f_{1,2} \circ r_{n+1} = I_{A/I}$, where the map $f_{1,2} : A/I^2 \longrightarrow A/I$ is the canonical projection. Let us set $u_2 = r_{n+1}$, the first step is ended; there exists a Poisson homomorphism of Poisson algebras $u_2 : A/I \longrightarrow A/I^2$ such that $f_{1,2} \circ u_2 = \mathrm{Id}_{A/I}$.

Second step. Step by step for $h \geq 2$, using Theorem 15, we have the diagram

$$\begin{tikzcd} 0 \arrow[r] & I^h/I^{h+1} \arrow[r] & A/I^{h+1} \arrow[r, "f_{h,h+1}"] & A/I^h \arrow[r] & 0 \\ & & & A/I \arrow[u, "u_h"'] \arrow[ul, "u_{h+1}"] & \end{tikzcd}$$

where the module I^h/I^{h+1} is a Poisson A/I^h module. Since the module I^h/I^{h+1} is a Poisson A/I module via u_h, there exists u_{h+1} homomorphism of Poisson algebras such that

$$f_{h,h+1} \circ u_h = u_{h+1}.$$

We conclude to the existence of the map $\hat{\varphi}: A/I \longrightarrow \widehat{A}$ such that $\widehat{\varphi}(x) = (x, u_2(x), u_3(x), \dots, u_h(x), \dots)$. The map $\widehat{\varphi}$ is a homomorphism of Poisson algebras satisfying $f_1 \circ \widehat{\varphi} = \mathrm{Id}_{A/I}$. □

2.3. Uniqueness of the lifting homomorphism from A/I into $\widehat{A}$

2.3.1. Study of a Poisson automorphism

Lemma 18. *Let I be a Poisson ideal of A generated by a centralizing sequence and $\widehat{A} = \varprojlim_n A/I^n$. Then for all $i \in \widehat{I}$, $\mathrm{e}^{\mathrm{ad}\, i} = \sum_{k=0}^{\infty} 1/k!\, \mathrm{ad}^k\, i$ defines a Poisson automorphism of Poisson algebras $\widehat{A}$, leaving stable $\hat{I}$ and acting trivially on $\widehat{A}/\hat{I}$.*

Proof. The map $\mathrm{e}^{\mathrm{ad}\, i}$ is well defined, since for all $x \in \widehat{A}$ and $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $\mathrm{ad}^k\, i(x) \in (I^n)^\wedge$; It is sufficient to use Corollary 12: $\forall n \in \mathbb{N} \quad \exists p \in \mathbb{N} : \mathfrak{D}_p I \subset I^n$. □

2.3.2. Uniqueness of the lifting homomorphism

Theorem 19. *Let A and I be a Poisson ideal A satisfying the lifting hypothesis (see Section 2.2). Let ρ', ρ'' be two homomorphisms of k-algebras from A/I into $\widehat{A}$ which are right inverse of $f_1 : \widehat{A} \longrightarrow A/I$. Then, there exists $i \in \widehat{I}$ such that $\rho' = \mathrm{e}^{\mathrm{ad}\, i} \circ \rho$.*

Proof. We know that if h is the length of the centralizing sequence generating I, then we have

$$\mathfrak{D}_{h+1} I + I^2 = I^2$$

(Corollary 12). We have the sequence of homomorphisms of Poisson algebras

$$\begin{aligned} &A/I^{n+1} \longrightarrow A/I^n \cdots \longrightarrow A/I^3 \longrightarrow A/I^2 \\ &\quad \longrightarrow A/\mathfrak{D}_h + I^2 \longrightarrow A/\mathfrak{D}_{h-1} I + I^2 \cdots \longrightarrow A/\mathfrak{D}_2 I + I^2 \\ &\quad \longrightarrow A/\mathfrak{D}_1 I + I^2 \longrightarrow A/I. \end{aligned}$$

Let us state $J_1 = I$, $J_2 = \mathfrak{D}_2 I + I^2, \dots, J_h = \mathfrak{D}_h I + I^2$, $J_{h+1} = I^2$, $J_{h+2} = I^3, \dots, J_{h+n} = I^{n+1}, \dots$

We have a projective limit, let us state $\widetilde{A} = \varprojlim_n A/J_n$. Let a be of A/I, the element $\rho(a) = (\rho_1(a), \rho_2(a), \dots, \rho_n(a), \dots)$ belongs to $\widehat{A} = \varprojlim_n A/I^n$. Fom $\rho(a)$, let us define $\widetilde{\rho}(a) = (\widetilde{\rho}_1(a), \widetilde{\rho}_2(a), \dots, \widetilde{\rho}_n(a), \dots)$ of the following form:

$$\begin{aligned} &\widetilde{\rho}_1(a) = \rho_1(a), \quad \widetilde{\rho}_2(a) = \varphi_2(\rho_2(a)), \dots, \\ &\widetilde{\rho}_h(a) = \varphi_h(\rho_2(a)), \quad \widetilde{\rho}_{h+1}(a) = \rho_2(a), \dots, \widetilde{\rho}_{h+n}(a) = \rho_{n+1}(a), \end{aligned}$$

where the map $\varphi_p : A/I^2 \longrightarrow A/\mathfrak{D}_p I + I^2$, for $2 \le p \le h$, is the canonical map. The element $\widetilde{\rho}(a)$ belongs to $\varprojlim_n A/J_n$.

Let $\widetilde{\rho} : A/J_1 \longrightarrow \varprojlim_n A/J_n$

and

$\widetilde{\rho}' : A/J_1 \longrightarrow \varprojlim_n A/J_n$ be obtained from ρ and ρ'.

In the following, we shall show the equality $\widetilde{\rho}' = \widetilde{\phi} \circ \widetilde{\rho}$ for some Poisson automorphism $\widetilde{A}$ and we shall deduct $\rho' = \phi \circ \rho$. This demonstration is done step by step.

Let us state $\Delta_1 = \widetilde{\rho}' - \widetilde{\rho}$ and denote $P_n : \widetilde{A} \longrightarrow A/J_n$ the canonical projection. For all a to A/I, it appears that $P_1 \circ \Delta_1(a) = 0$ denoting $\widetilde{J}_n = \operatorname{Ker} P_n$, for $n \in \mathbb{N}$; thus the map Δ_1 satisfies

$$\Delta_1 : A/J_1 \longrightarrow \widetilde{J}_1.$$

Let us show that $P_2 \circ \Delta_1 : A/J_1 \longrightarrow J_1/J_2$ defines a Poisson 1-cocycle, in other words $P_2 \circ \Delta_1$ belongs to $Z^1_{\text{Poisson}}(A/J_1, J_1/J_2)$. The module J_1/J_2 is an A/J_1-module by

$$a \cdot j = P_2(\widetilde{\rho}(a)) \cdot j \quad \text{et} \quad [a, j] = \{P_2(\widetilde{\rho}(a)), j\}_{A/J_2} a \in A/J_1, \quad j \in J_1/J_2;$$

a simple calculation shows that $(d_1(P_2 \circ \Delta_1))(a, b) = -\{P_2 \circ \Delta_1(a), P_2 \circ \Delta_1(b)\}$. But $P_2 \circ \Delta_1(a)$ belongs to J_1/J_2 and $\{J_1, J_2\}$ is included in J_2. Thus we have

$$d_1(P_2 \circ \Delta_1) = 0,$$

and thus $P_2 \circ \Delta_1$ belongs to $Z^1_{\text{Poisson}}(A/J_1, J_1/J_2)$. The module J_1/J_2 is generated by a centralizing sequence, Theorem 16 leads to the isomorphism $J_1/J_2 \simeq A/J_1 \otimes H^0_{\text{Poisson}}(A/J_1, J_1/J_2)$. But the equality $H^1_{\text{Poisson}}(A/J_1, A/J_1) = 0$ (Proposition 10), implies $H^1_{\text{Poisson}}(A/J_1, J_1/J_2) = 0$. The map $P_2 \circ \Delta_1$ is therefore a coboundary; this implies

$$P_2 \circ \Delta_1(a) = [a, u_1],$$

for all a of A/J_1 and for some $u_1 \in J_1/J_2$. From this, we deduct that $\Delta_1(a) - \{v_1, \widetilde{\rho}(a)\}$ belongs to $\operatorname{Ker} P_2 = \widetilde{J}_2$ for some $v_1 \in \widetilde{J}_1$. Since $v_1 \in \widetilde{J}_1$, according to Theorem 18 $\mathrm{e}^{\operatorname{ad} v_1}$ is a Poisson automorphism from $\widetilde{A}$ onto $\widetilde{A}$. We have

$$\widetilde{\rho}'(a) \equiv \mathrm{e}^{\operatorname{ad} v_1} \widetilde{\rho}(a), \quad \text{modulo } \widetilde{J}_2.$$

Let us denote $\widetilde{\rho}_1 = \mathrm{e}^{\operatorname{ad} v_1} \widetilde{\rho}$.

$\Delta_2 = \widetilde{\rho}' - \widetilde{\rho}_1$, then the element $P_2 \circ \Delta_2(a)$ belongs to $\operatorname{Ker} P_2 = \widetilde{J}_2$.

Following step by step, at the end of a first stage

$$\widetilde{\rho}'(a) \equiv \mathrm{e}^{\operatorname{ad} v_h} \mathrm{e}^{\operatorname{ad} v_{h-1}} \cdots \mathrm{e}^{\operatorname{ad} v_1} \widetilde{\rho}(a), \quad \text{modulo } \widetilde{J}_{h+1} = (I^2)^\wedge,$$

where $v_1 \in \widetilde{J}_1, \ldots, v_h \in \widetilde{J}_h$. We continue

$$\widetilde{\rho}'(a) \equiv \mathrm{e}^{\operatorname{ad} v_{n+h}} \cdots \mathrm{e}^{\operatorname{ad} v_{1+h}} \cdots \mathrm{e}^{\operatorname{ad} v_1} \widetilde{\rho}(a), \quad \text{modulo } \widetilde{J}_{n+1+h}.$$

The $e^{\mathrm{ad}\, v_{l+h}}$ are Poisson automorphisms $\widetilde{A}$ from Theorem 18. Let us set $\widetilde{\phi}_e = e^{\mathrm{ad}\, v_e} \cdots e^{\mathrm{ad}\, v_1}$, it is a Poisson automorphism $\varprojlim_n A/J_n$ by composition. For all $a \in A/J_1$ the sequence $(\widetilde{\phi}_l(a))_{l\in\mathbb{N}}$ in $\widetilde{A}$ is a Cauchy sequence in $\widetilde{A}$. Hence the sequence $(\widetilde{\phi}_l(a))_{l\in\mathbb{N}}$ is convergent in $\widetilde{A}$ separated complete. Let us denote $\widetilde{\phi}(a) = \lim_{l\longrightarrow\infty}\widetilde{\phi}_l(a)$, $\widetilde{\phi}$ belongs to $\mathrm{Aut}_{\mathrm{Poisson}}\, \widetilde{A}$. Finally, we have found $\widetilde{\phi} : \widetilde{A} \longrightarrow \widetilde{A}$ Poisson automorphism such that $\widetilde{\rho}' = \widetilde{\phi}\circ\widetilde{\rho}$. If we restrict ourselves to the ideals I^n, from this we deduct the existence of the map ϕ of $\mathrm{Aut}_{\mathrm{Poisson}}(\widehat{A})$ such that $\rho' = \phi \circ \rho$. □

Corollary 20. *Let us assume that the hypotheses of Theorem* 19 *are satisfied. Let C be the commutant associated to a lifting Poisson homomorphism from A/I into $\widehat{A}$.*

Then the algebra C is independent from the choice of the lifting homomorphism (modulo Poisson isomorphisms).

2.4. Study of the commutant

2.4.1. Properties related to the completion

We shall call a filtration of a Poisson algebra A, a decreasing filtration of A given by vector subspace $(A_n)_{n\in\mathbb{Z}}$ such that

1. $A_n A_m \subset A_{n+m}\ \forall n, m \in \mathbb{Z}$
2. $\{A_n, A_m\} \subset A_{n+m-1}\ \forall n, m \in \mathbb{Z}$.

Let I a Poisson ideal A, the *I-adic filtration of A* is defined by $A_n = I^n$, $n \geq 0$ and where $I^n = A$ for $n \leq 0$.

If A is filtered by $(A_n)_{n\in\mathbb{Z}}$, we shall call filtration of a Poisson A-module M, a decreasing filtration of M for the structure of A-module by subspaces $(M_n)_{n\in\mathbb{Z}}$ with the additional condition

$$A_n M_m \subset M_{n+m} \quad \text{et} \quad [A_n, M_m] \subset M_{n+m-1}, \quad n, m \in \mathbb{Z}.$$

The *I-adic filtration of M* is defined by: $M_n = I^n M$.

Let A be a Poisson algebra and I a Poisson ideal. We endow A with the topology associated to the I-adic filtration of A. A completion of A is a filtered separated Poisson algebra $\tilde{A}$ complete for the associated topology, provided with a homomorphism of Poisson algebras $j_A : A \longrightarrow \tilde{A}$ continuous and satisfying: for any filtered separated complete Poisson algebra A' and any continuous homomorphism of Poisson algebras $f : A \longrightarrow A'$, there exists a unique and continuous homomorphism of Poisson algebras $\tilde{f} : \tilde{A} \longrightarrow A'$ such that $\tilde{f} \circ j_A = f$. A completion of I-adic is unique up to Poisson isomorphism, hence we shall say the completion of A.

We choose $\widehat{A} = \varprojlim_n A/I^n$ filtered by $(I^n)^\wedge = \mathrm{Ker}(\widehat{A} \longrightarrow A/I^n)$ and the map $j_A : A \longrightarrow \widehat{A}$ defined by $j_A(x) = (x+I, x+I^2, \ldots)$. We have $j_A(I^n) = j_A(A) \cap \mathrm{Ker}(\widehat{A} \longrightarrow A/I^n)$ and the image $j_A(A)$ is dense in $\widehat{A}$.

Likewise, let M be a Poisson A-module endowed with the I-adic filtration, we define a completion of M denoted $\widehat{M}$. Then a completion is unique up to Poisson isomorphism.

We take $\widehat{M} = \varprojlim_n M/I^n M$, filtered by the sets $\mathrm{Ker}(\widehat{M} \longrightarrow M/I^n M)$. We denote by $j_M : M \longrightarrow \widehat{M}$ the map defined by $j_M(m) = (m + IM, m + I^2 M, \ldots)$, it is a homomorphism of Poisson A-module. We have $j_M(I^n M) = j_M(M) \cap \mathrm{Ker}(\widehat{M} \longrightarrow M/I^n M)$ for $n \geq 1$, and the image $j_M(M)$ is dense in $\widehat{M}$. Moreover, $\widehat{M}$ is a Poisson $\widehat{A}$-module such that for all a in A and all $\widehat{m}$ of $\widehat{M}$ we have $j_A(a)\widehat{m} = a \cdot \widehat{m}$ et $[j_A(a), \widehat{m}] = [a, \widehat{m}]$. From this equality, we deduct $j_M(a \cdot m) = j_A(a) j_M(m)$ et $j_M([a, m]) = [j_A(a), j_M(m)]$ for a in A and m in M.

Lemma 21. *Let I be a Poisson ideal generated by a centralizing sequence $(x_1, \ldots, x_n)$ of A. Let us assume A is notherian. Then*
(1) *Let us set $\overline{A} = A/Ax_1 + \cdots + Ax_{i-1}$ for $i \geq 2$ and $\overline{I} = \overline{A}\overline{x_n} + \cdots + \overline{A}\overline{x_i}$, the $\overline{I}$-adic completion of $\overline{A}$, denoted by $\widehat{\overline{A}}$, is isomorphic as Poisson algebra to $\widehat{A}/\widehat{A} j_A(x_1) + \cdots + \widehat{A} j_A(x_{i-1})$.*
(2) *We have $\widehat{I} = \widehat{A} j_A(x_1) + \cdots + \widehat{A} j_A(x_n)$ and the sequence $(j_A(x_1), \ldots, j_A(x_n))$ is centralizing in $\widehat{A}$.*

If the sequence $(x_1, \ldots, x_n)$ is regular in A, then $(j_A(x_1), \ldots, j_A(x_n))$ is a regular sequence in $\widehat{A}$.

Proof. See [31, p. 41]. □

2.4.2. Properties of the commutant associated to a lifting homomorphism from A/I into $\varprojlim A/I^n$

Lemma 22. *Let A and I satisfy the lifting hypothesis and φ a lifting homomorphism from A/I into $\widehat{A}$. Let $\widehat{I}$ be the ideal $\mathrm{Ker}(\widehat{A} \longrightarrow A/I)$ of $\widehat{A}$ and C the commutant of $\varphi(A/I)$ in $\widehat{A}$. Let us denote m the ideal $\widehat{I} \cap C$.*

Then
1. *We have an isomorphism of A/I-modules*

$$(I^n)^\wedge \cap C/(I^{n+1})^\wedge \cap C \simeq H^0_{\mathrm{Poisson}}(A/I, I^n/I^{n+1}).$$

2. *We have the equality*

$$m^n = (I^n)^\wedge \cap C.$$

Proof.
(1) We have the exact sequence of Poisson A/I-module for $n \geq 0$:

$$0 \longrightarrow (I^{n+1})^\wedge \longrightarrow (I^n)^\wedge \longrightarrow (I^n)^\wedge/(I^{n+1})^\wedge \longrightarrow 0.$$

We deduct the exact sequence

$$0 \longrightarrow H^0_{\mathrm{Poisson}}(A/I, (I^{n+1})^\wedge) \longrightarrow H^0_{\mathrm{Poisson}}(A/I, (I^n)^\wedge) \longrightarrow H^0_{\mathrm{Poisson}}(A/I, (I^n)^\wedge/(I^{n+1})^\wedge),$$

which is the sequence

$$0 \longrightarrow (I^{n+1})^\wedge \cap C \longrightarrow (I^n)^\wedge \cap C \longrightarrow H^0(A/I, (I^n)^\wedge/(I^{n+1})^\wedge).$$

Let us show that the last arrow of this sequence corresponds to a surjective map. Let y be in $H^0(A/I, (I^n)^\wedge/(I^{n+1})^\wedge)$. We have $y = c_n + (I^{n+1})^\wedge$ for some $c_n \in (I^n)^\wedge$ and $\{\varphi(a), c_n\} \in (I^{n+1})^\wedge$, $\forall a \in A/I$. The map $\zeta : A/I \longrightarrow (I^{n+1})^\wedge/(I^{n+2})^\wedge$ defined by $\zeta(a) = \{\varphi(a), c_n\} + (I^{n+2})^\wedge$ is a cocycle, that is to say an element in $Z^1_{\text{Poisson}}(A/I, (I^{n+1})^\wedge/(I^{n+2})^\wedge)$. From the isomorphism $(I^k)^\wedge/(I^{k+1})^\wedge \simeq I^k/I^{k+1}$ for all $k \in \mathbb{N}$, we can see that the A/I-module $(I^{n+1})^\wedge/(I^{n+2})^\wedge$ is generated by a centralizing sequence. From Theorem 16, we have the isomorphism $(I^{n+1})^\wedge/(I^{n-2})^\wedge \simeq A/I \otimes H^0(A/I, (I^{n+1})^\wedge/(I^{n+2})^\wedge)$. Consequently the module $(I^{n+1})^\wedge/(I^{n+2})^\wedge$ is a sum of copies of A/I; from the equality $H^1_{\text{Poisson}}(A/I, A/I) = 0$ (Proposition 10), we obtain

$$H^1_{\text{Poisson}}(A/I, (I^{n+1})^\wedge/(I^{n+2})^\wedge) = 0.$$

Therefore the map ζ is a coboundary. There exists an element b_{n+1} of $(I^{n+1})^\wedge$ such that we have $\{\varphi(a), -b_{n+1}\} \equiv \{\varphi(a), c_n\}$ modulo $(I^{n+2})^\wedge$ for all a in A/I. Hence we have

$$\{\varphi(a), c_{n+1}\} \in (I^{n+2})^\wedge, \quad c_{n+1} = c_n + b_{n+1}, \quad b_{n+1} \in (I^{n+1})^\wedge.$$

Step by step, we obtain a sequence of elements $(c_{n+k})_{k\in\mathbb{N}}$ in $(I^{n+k})^\wedge$ such that $\{\varphi(a), c_{n+k}\} \in (I^{n+k+1})^\wedge$, $c_{n+k} = c_{n+k-1} + b_{n+k}$, where b_{n+k} belongs to $(I^{n+k})^\wedge$. This Cauchy sequence of $(I^n)^\wedge$ converges to an element x in $(I^n)^\wedge$ satisfying

$$\{\varphi(a), x\} = 0, \quad \forall a \in A/I, \quad x \equiv y, \quad \operatorname{mod}(I^{n+1})^\wedge.$$

Thus, the sequence

$$0 \longrightarrow (I^{n+1})^\wedge \cap C \longrightarrow (I^n)^\wedge \cap C \longrightarrow H^0(A/I, I^n/I^{n+1}) \longrightarrow 0$$

is exact. We conclude that

$$(I^n)^\wedge \cap C/(I^{n+1})^\wedge \cap C \simeq H^0_{\text{Poisson}}(A/I, I^n/I^{n+1})$$

for $n \geq 0$.

(2) The inclusion $(\widehat{I})^n \subset (I^n)^\wedge$ implies $m^n \subset (I^n)^\wedge \cap C$. To check the inverse inclusion, we use the isomorphism of Lemma 22 $(I^n)^\wedge \cap C/(I^{n+1})^\wedge \cap C \simeq H^0_{\text{Poisson}}(A/I, I^n/I^{n+1})$. Let us consider the surjective map $\mu : I/I^2 \otimes_{A/I} \cdots \otimes_{A/I} I/I^2$ $(n\ times) \longrightarrow I^n/I^{n+1}$ obtained by using the multiplication in I-adic graduate of A. The sets I/I^2 being Poisson A/I-modules via φ, the same happened for $I/I^2 \otimes_{A/I} \cdots \otimes_{A/I} I/I^2$ from Property 5; we verify that μ is a homomorphism of Poisson A/I-modules. We have the exact sequence of Poisson A/I-modules

$$0 \longrightarrow \operatorname{Ker}\mu \longrightarrow I/I^2 \otimes_{A/I} \cdots \otimes_{A/I} I/I^2 \ (n\ times) \xrightarrow{\mu} I^n/I^{n+1} \longrightarrow 0.$$

From this, we deduct the exact sequence

$$
\begin{aligned}
&0 \longrightarrow H^0_{\text{Poisson}}(A/I, \operatorname{Ker}\mu) \longrightarrow H^0_{\text{Poisson}}(A/I, I/I^2 \otimes_{A/I} \cdots \otimes_{A/I} I/I^2) \xrightarrow{H^0(\mu)} \\
&H^0_{\text{Poisson}}(A/I, I^n/I^{n+1}) \xrightarrow{\omega_0} H^1_{\text{Poisson}}(A/I, \operatorname{Ker}\mu) \longrightarrow \\
&H^1_{\text{Poisson}}(A/I, I/I^2 \otimes_{A/I} \cdots \otimes_{A/I} I/I^2) \xrightarrow{H^1(\mu)} H^1_{\text{Poisson}}(A/I, I^n/I^{n+1}) \longrightarrow \cdots
\end{aligned}
$$

The map ω_0 is said to be the connecting homomorphism. We know according to Theorem 16 that I/I^2 is a direct sum of copies of A/I, and thus too $I/I^2 \otimes_{A/I} \cdots \otimes_{A/I} I/I^2$. But the algebra A/I is a simple A/I-module. One more time the A/I-module $\operatorname{Ker}\mu$, submodule of the semi-simple module $I/I^2 \otimes_{A/I} \cdots \otimes_{A/I} I/I^2$ is a direct sum of copies of A/I. The equality $H^1_{\text{Poisson}}(A/I, A/I) = 0$ (Proposition 10) implies the following $H^1_{\text{Poisson}}(A/I, \operatorname{Ker}\mu) = 0$. Thus the sequence

$$
\begin{aligned}
&H^0_{\text{Poisson}}(A/I, I/I^2) \otimes \cdots \otimes H^0_{\text{Poisson}}(A/I, I/I^2) \\
&\quad \xrightarrow{H^0(\mu)} H^0_{\text{Poisson}}(A/I, I^n/I^{n+1}) \longrightarrow 0
\end{aligned}
$$

is exact. From the surjectivity of $H^0(\mu)$, we infer the inclusion $(I^n)^\wedge \cap C \subset (\widehat{I} \cap C)^n$, id est $(I^n)^\wedge \cap C \subset m^n$. $\square$

Theorem 23. *Let A and I satisfy the lifting hypothesis (Section 2.2). Let us denote φ a lifting homomorphism from A/I in $\varprojlim_n A/I^n = \widehat{A}$. Let us consider C the commutant of $\varphi(A/I)$ in $\widehat{A}$ and let $m = \widehat{I} \cap C$. Then*

1. *The commutant C is a closed Poisson subalgebra and a local ring of maximal ideal m, of residue field k satisfying*

$$C = \varprojlim_n C/m^n.$$

2. *Let $(e_n)_{n\in\mathbb{N}}$ be a sequence of elements of C such that e_0+m is C/m, $e_1+m^2, \ldots, e_{\alpha(1)}+m^2$ a basis of m/m^2, $e_{\alpha(p-1)+1} + m^{p+1}, \ldots, e_{\alpha(p)} + m^{p+1}$ a basis of m^p/m^{p+1} for $p \geq 2$.*

   Then

   – *For all $c \in C$ there exists a unique sequence $(c_p)_{p\in\mathbb{N}}$ of k such that*

$$c = \sum c_p e_p.$$

   – *For all a of $\widehat{A}$ there exists a unique sequence $(a_p)_{p\in\mathbb{N}}$ of $\varphi(A/I)$ such that*

$$a = \sum a_p e_p.$$

3. *Moreover, let us assume that A is noetherian and I generated by a regular centralizing sequence $(x_1, \ldots, x_n)$, then the maximal ideal of C is of form $m = Cj_A(x_1) + Cy_2 + \cdots + Cy_n$, where $(j_A(x_1), y_2, \ldots, y_n)$ is a regular centralizing sequence, $j_A : A \longrightarrow \widehat{A}$ being the canonical homomorphism. The algebra C is isomorphic as an associative*

commutative unitary algebra to the formal power series algebra $k[[X_1, \dots, X_n]]$, n being the length of the regular centralizing sequence generating I.

Proof.

(1) To show that C is a local ring with maximal ideal $m = \widehat{I} \cap C$, it is sufficient to prove that $m \neq C$ and that any element of $C - m$ is a unit in C. The unit of $\widehat{A}$ does not belong to $\widehat{I}$, this implies that $\widehat{I} \cap C \neq C$. Let $\widehat{x}$ be an element in $C \setminus (C \cap \widehat{I})$, then $\widehat{x}$ is like $(x_1, \dots, x_n, \dots)$ with $x_1 \neq 0$ and $\{\widehat{x}, \varphi(t)\} = 0$ for all $t \in A/I$. We apply the map $f_1 : \widehat{A} \longrightarrow A/I$ to this last relation that we know $f_1 \circ \varphi = \mathrm{Id}_{A/I}$ and $f_1(\widehat{x}) = x_1$; thus x_1 belongs to $C(A/I)$ which is isomorphic to k as the center of a Poisson–Weyl algebra. Hence the element $x_1 \neq 0$ is invertible in A/I. We deduct step by step that $\widehat{x}$ is invertible in $\widehat{A}$. Let $\widehat{y}$ be its inverse. From $\widehat{x}\widehat{y} = 1$, we see that we have, for all $t \in A/I$, $1 \cdot \{\widehat{y}, \varphi(t)\} = 0$; thus $\widehat{y}$ belongs to C.

Let us verify that the field $C/C \cap \widehat{I}$ is k. We have $C/C \cap \widehat{I} = C + \widehat{I}/\widehat{I}$ by the second isomorphism theorem of Poisson algebras. We have the injective map $C + \widehat{I}/\widehat{I} \longrightarrow \widehat{A}/\widehat{I}$, but $\widehat{A}/\widehat{I}$ is isomorphic to A/I and since A/I is isomorphic to a Poisson–Weyl algebra by hypothesis, it follows that: $C/C \cap \widehat{I} \subset k$.

From Lemma 22 $m^n = \widehat{I^n} \cap C$, we deduct that C is complete for the m-adic topology: $C = \varprojlim_n C/m^n$.

(2) – For all $l \in \mathbb{N}$, m^l/m^{l+1} is a vector space over $C/m = k$ finite-dimensional. Let $c \in C$, then $c + m = c_0(e_0 + m)$ for some c_0 in k. The class modulo m^2 of the element in m, $c - c_0 e_0$, has a unique representation on the basis $e_1 + m^2, \dots, e_{\alpha(1)} + m^2$ of m/m^2. Therefore, step by step, for all $p \in \mathbb{N}$, there exists a unique sequence of elements $c_0, \dots, c_{\alpha(p)}$ in k such that

$$c - \sum_{l=0}^{l=\alpha(p)} c_l e_l \in m^{p+1}.$$

Thus we can construct a unique sequence of elements $(c_j)_{j \in \mathbb{N}}$ in k such that $u_i = c - \sum_{l=0}^{l=i} c_l e_l$ converges to 0 for the m-adic topology of C.

– Let $a \in \widehat{A}$, to approach a, we shall prove the following isomorphism:

$$(I^n)^\wedge/(I^{n+1})^\wedge \simeq A/I \otimes m^n/m^{n+1}.$$

From Lemma 22, we have therefore the isomorphism $m^n/m^{n+1} \simeq H^0_{\text{Poisson}}(A/I, I^n/I^{n+1})$. We know that $(I^n)^\wedge/(I^{n+1})^\wedge \simeq I^n/I^{n+1}$. The module I^n/I^{n+1} is a Poisson A/I-module generated by a centralizing sequence and the algebra is A/I a Poisson–Weyl algebra; thus from Theorem 16, we have the isomorphism $I^n/I^{n+1} \simeq A/I \otimes_k H^0_{\text{Poisson}}(A/I, I^n/I^{n+1})$. It follows that $(I^n)^\wedge/(I^{n+1})^\wedge \simeq A/I \otimes m^n/m^{n+1}$. By using this last isomorphism with the same argument as the one used previously, we construct a unique sequence $(a_p)_{p \in \mathbb{N}}$ of elements in $A/I \simeq \varphi(A/I)$ such that

$$a = \sum a_p e_p.$$

(3) We proceed by induction on the length n of the regular centralizing sequence $(x_1, \dots, x_n)$.

For $n = 1$, $I = Ax_1$. From Lemma 21, the ideal $\widehat{I}$ is $\widehat{A}j_A(x_1)$. The element x_1 being central and regular, also is $j_A(x_1)$, the algebra A being noetherian. We check that in this case we have $\widehat{I} \cap C = Cj_A(x_1)$. We conclude that if $I = Ax_1$, then the maximal ideal of C is $m_1 = Cj_A(x_1)$, generated by a regular centralizing sequence. Let us assume that I is generated by a regular centralizing sequence of length n, $I = Ax_1 + \cdots + Ax_n$. Let us state $\overline{A} = A/Ax_1$, $\overline{I} = I/Ax_1$ and $\widehat{\overline{A}}$ the $\overline{I}$-adic completion of $\overline{A}$. The ideal $\overline{I} = \overline{A}\overline{x_2} + \cdots + \overline{A}\overline{x_n}$ is generated by a regular centralizing sequence of $\overline{A}$ with a length $n - 1$. From Lemma 21, the $\overline{I}$-adic completion of $\overline{A}$ denoted by $\widehat{\overline{A}}$ is isomorphic to $\widehat{A}/\widehat{A}j_A(x_1)$, $\widehat{A}$ being the I-adic completion of A. Let us denote by $p : \widehat{A} \longrightarrow \widehat{A}/\widehat{A}j_A(x_1)$ the canonical projection. We notice that $\overline{A}/\overline{I} = (A/Ax_1)/(I/Ax_1)$ is isomorphic to A/I using the first isomorphism theorem of Poisson algebras.

Let the diagram

$$\begin{tikzcd}
\overline{A}/\overline{I} \simeq A/I \arrow[r, "\varphi"] \arrow[dr, "p\circ\varphi"'] & \widehat{A} \arrow[d, "p"] \\
 & \widehat{A}/\widehat{A}j_A(x_1)
\end{tikzcd}$$

Using the isomorphism

$$A/I \simeq \overline{A}/\overline{I},$$

we check that the map $p \circ \varphi$ is a lifting homomorphism $\overline{\pi} : \widehat{\overline{A}} \longrightarrow A/I$:

$$p \circ \varphi \circ \overline{\pi} = \mathrm{Id}_{\overline{A}/\overline{I}}.$$

The lifting homomorphism from A/I into $\widehat{A}/\widehat{A}j(x_1)$ is $p \circ \varphi(A/I)$. Let us denote by $\overline{C}$ the commutant of $p \circ \varphi(A/I)$ in $\widehat{A}/\widehat{A}j_A(x_1)$. We check that $\overline{C} = C/Cj_A(x_1)$ from the following lemma, which is further proved. From (1) the algebra $\overline{C}$ is a local ring. Its maximal ideal denoted by $\overline{m}$ is $m/Cj_A(x_1)$; ad indeed $(C/Cj_A(x_1))/(m/Cj_A(x_1))$ is isomorphic to C/m and (C/m) is isomorphic to k. Thus we apply the lifting hypothesis of induction to $\overline{m} = \overline{A}\overline{x_2} + \cdots + \overline{A}\overline{x_n}$. The ideal $\overline{m}$ is of length $n - 1$ and is generated by a regular centralizing sequence of $(y_2, \dots, y_n)$ de $\overline{C}$:

$\overline{m} = \overline{C}\overline{y_2} + \cdots + \overline{C}\overline{y_n}$. Thus we have

$$m = Cj(x_1) + Cy_2 + \cdots + Cy_n,$$

where the sequence $(j_A(x_1), y_2, \dots, y_n)$ is regular and centralizing.

From a classical result (see [31, p. 48]), there exists a unique isomorphism of unitary algebras $\widehat{\varphi} : k[[X_1, \dots, X_n]] \longrightarrow C$. □

Lemma 24. *Under the hypotheses of Theorem* 23 *and its notations, the sequence*

$$0 \longrightarrow Cj_A(x_1) \longrightarrow C \overset{p}{\longrightarrow} \overline{C} \longrightarrow 0$$

is exact.

Proof. The algebra $\widehat{A}$ (respectively $\widehat{A}/\widehat{A}j_A(x_1)$) are Poisson A/I-module via φ (respectively $p \circ \varphi$). From the short exact sequence $0 \longrightarrow \widehat{A}j_A(x_1) \longrightarrow \widehat{A} \longrightarrow \widehat{A}/\widehat{A}j_A(x_1) \longrightarrow 0$, we infer the long exact sequence $0 \longrightarrow H^0_{\text{Poisson}}(A/I, \widehat{A}j_A(x_1)) \longrightarrow H^0_{\text{Poisson}}(A/I, \widehat{A}) \longrightarrow H^0_{\text{Poisson}}(A/I, \widehat{A}/\widehat{A}j_A(x_1)) \longrightarrow H^1_{\text{Poisson}}(A/I, \widehat{A}j_A(x_1)) \longrightarrow \cdots$ Hence it is enough to prove that $H^1_{\text{Poisson}}(A/I, \widehat{A}j_A(x_1)) = 0$; it is obtained from the following facts:

The ideal $\widehat{A}j_A(x_1)$ is equal to $\widehat{A}$ while the element $j_A(x_1)$ is regular. From the assertion of Theorem 23, the algebra $\widehat{A}$ is a product of copies A/I; but we have $H^1_{\text{Poisson}}(A/I; A/I) = 0$ from Proposition 10, and we use the general commutation formula $\text{Ext}_\Lambda(B, \prod_j M_j) \simeq \prod \text{Ext}_\Lambda(B, M_j)$.

We have the equalities $H^0_{\text{Poisson}}(A/I, \widehat{A}j_A(x_1)) = Cj_A(x_1)$, $H^0_{\text{Poisson}}(A/I, \widehat{A}) = C$, $H^0_{\text{Poisson}}(A/I; \widehat{A}/\widehat{A}j_A(x_1)) = \overline{C}$. Thus the sequence $0 \longrightarrow Cj_A(x_1) \longrightarrow C \longrightarrow \overline{C} \longrightarrow 0$ is exact. □

2.5. *Study of the graded spaces*

2.5.1. *Properties of the I-adic graded algebras*

Let A be a Poisson algebra and I a Poisson ideal. The associated graded I-adic algebra $\text{Gr}_I A = \oplus_{n \geq 0} I^n/I^{n+1}$ is a Poisson algebra endowed with the operations such that

$$\pi_n(a)\pi_m(b) = \pi_{m+n}(ab), \quad \text{and} \quad \{\pi_n(a), \pi_m(b)\} = \pi_{m+n-1}(\{a, b\})$$

for $n, m \geq 0$, $a \in I^n$ and $b \in I^m$, where $\pi_n : I^n \longrightarrow I^n/I^{n+1}$ is the canonical projection et $\pi_{-1} = 0$ $(I_{-1} = I_0 = A)$.

We notice that in the Poisson algebra $\text{Gr}_I A$, we have for all $n \geq 0$ $\{A/I, I^n/I^{n+1}\} = 0$. Ad indeed, for all $n \geq 0$ and for all x in I^n, we have the relations $\{\pi_0(a), \pi_n(x)\} = \pi_{n-1}(\{a, x\})$ and $\{A, I^n\} \subset I^n$ which implies $\{\pi_0(a), \pi_n(x)\} = 0,\ n \geq 0$.

We know that I/I^2 is a Lie A/I-algebra. Therefore the symmetric algebra $S_{A/I}(I/I^2)$ is a Poisson algebra. The injective map $i : I/I^2 \longrightarrow \text{Gr}_I A$ is extending in a unique way by universal property to a Poisson algebra homomorphism from the symmetric algebra $S_{A/I}(I/I^2)$ into $\text{Gr}_I A$. This homomorphism is surjective while all element of I^n/I^{n+1} is a product of n elements in I/I^2. We have the exact sequence of Poisson algebras $S_{A/I}(I/I^2) \longrightarrow \text{Gr}_I A \longrightarrow 0$.

If we assume the existence of a Poisson homomorphism $\varphi : A/I \longrightarrow \widehat{A}$, the I-adic graded algebras $\text{Gr}_I A$ may be provided with a Poisson structure of Poisson A/I-module via φ.

Property 25. *Let $f_n : \widehat{A} \longrightarrow A/I^n$ be the canonical projection and $\pi_n : I^n \longrightarrow I^n/I^{n+1}$. We suppose there exists a homomorphism of Poisson algebras φ from A/I into $\widehat{A}$. Then*

(1) *The algebra $\text{Gr}_I A$ is a Poisson A/I-module defined by, $\forall a \in A, \forall n \geq 0, \forall x \in I^n$:*

$$\pi_0(a).\pi_n(x) = f_{n+1}(\varphi(\pi_0(a)))\pi_n(x),$$

et

$$[\pi_0(a), \pi_n(x)] = \{f_{n+1}(\varphi(\pi_0(a))), \pi_n(x)\}_{A/I^{n+1}}.$$

(2) *Let us assume* $\pi_0(a) = \overline{a}$, $a \in A$, *we have the following formulas in the* A/I*-module* $\mathrm{Gr}_I A$: $\forall x, \forall y \in \mathrm{Gr}_I A$ *et* $\forall a \in A$:

$$[\overline{a}, xy] = x[\overline{a}, y] + [\overline{a}, x]y,$$

et

$$[\overline{a}, \{x, y\}_{\mathrm{Gr}_I A}] = \{[\overline{a}, x], y\} + \{x, [\overline{a}, y]\}_{\mathrm{Gr}_I A}.$$

Proof.

(1) It is a simple verification.

(2) See [31, p. 50]. □

Corollary 26. *Under the hypotheses of Property* 25, *we provide* $\mathrm{Gr}_I A$ *with the structure of Poisson* A/I*-module via* φ.

Then $H^0_{\mathrm{Poisson}}(A/I, \mathrm{Gr}_I A)$ *is a Poisson subalgebra of* $\mathrm{Gr}_I A$.

2.5.2. Decomposition of the I-adic graded algebra and gradation of the commutant in its maximal ideal

We see that the Poisson algebra A/I, apart from the structure of quotient Poisson algebra, has a Poisson structure induced by this of $\mathrm{Gr}_I A$ which is trivial.

Theorem 27 (Decomposition of the graded algebra). *Let A and I verify the lifting hypothesis (Section 2.2). We choose a lifting homomorphism from* A/I *into* $\widehat{A}$ *and provide* $\mathrm{Gr}_I A$ *with the Poisson structure of* A/I*-module via* φ. *Let us denote C the commutant of* $\varphi(A/I)$ *in* $\widehat{A}$, $m = \widehat{I} \cap C$ *the maximal ideal of C and* $\mathrm{Gr}_m C = \oplus_{n\geq 0} m^n/m^{n+1}$ *the m-adic graded algebra of C. We provide the algebra* $A/I \otimes H^0_{\mathrm{Poisson}}(A/I, \mathrm{Gr}_I A)$ *with the structure of Poisson algebra extending the trivial Poisson of* A/I *and the Poisson structure of* $H^0_{\mathrm{Poisson}}(A/I, \mathrm{Gr}_I A)$. *Then*

1. *The map* $\phi : A/I \otimes H^0_{\mathrm{Poisson}}(A/I, \mathrm{Gr}_I A) \longrightarrow \mathrm{Gr}_I A$ *such that* $\phi(a \otimes t) = a\, t$, *for* $a \in A/I$ *and* $t \in \mathrm{Gr}_I A$, *is an isomorphism of graded Poisson algebras.*
2. *We have*

$$\mathrm{Gr}_m C \simeq H^0_{\mathrm{Poisson}}(A/I, \mathrm{Gr}_I A),$$
$$\mathrm{Gr}_{(\widehat{I} \cap C)}(H^0_{\mathrm{Poisson}}(A/I, \widehat{A})) \simeq H^0_{\mathrm{Poisson}}(A/I, \mathrm{Gr}_I A),$$
$$\mathrm{Gr}_I A \simeq A/I \otimes \mathrm{Gr}_m C.$$

Proof.

(1) Let us prove that the map ϕ is a homomorphism of Poisson algebras. Let $a, b \in A/I$ and $x, y \in \mathrm{Gr}_I A$, we have $\{a \otimes x, b \otimes y\} = \{a, b\} \otimes xy + ab \otimes \{x, y\}$ and we use the fact that $\{a, b\} = 0$, $\{a, x\} = \{b, y\} = 0$ in $\mathrm{Gr}_I A$. The map ϕ is an isomorphism. It is a consequence of Theorem 16. For all $n \geq 0$, the module I^n/I^{n+1} is a Poisson A/I-module generated by a centralizing sequence and A/I a Poisson–Weyl k-algebra; thus, we deduct the isomorphism $A/I \otimes H^0_{\mathrm{Poisson}}(A/I, I^n/I^{n+1}) \simeq I^n/I^{n+1}$. By property of the tensor product and commutation of the Poisson cohomology with the direct sums, we obtain

$$A/I \otimes H^0_{\mathrm{Poisson}}(A/I, \mathrm{Gr}_I A) \simeq \mathrm{Gr}_I A.$$

(2) By virtue of Lemma 22, we see that $(I^n)^\wedge \cap C/(I^{n+1})^\wedge \cap C \simeq H^0_{\mathrm{Poisson}}(A/I, I^n/I^{n+1})$. But we have the equality $(I^n)^\wedge \cap C = m^n$ and therefore the isomorphism $m^n/m^{n+1} \simeq H^0(A/I, I^n/I^{n+1})$ for $n \geq 0$.

By commutation of the Poisson cohomology with the direct sums, we conclude that $\mathrm{Gr}_m C = H^0(A/I, \mathrm{Gr}_I A)$. By using the item one, we have $\mathrm{Gr}_I A \simeq A/I \otimes \mathrm{Gr}_m C$. □

3. Lifting map and symmetric algebra of a nilpotent Lie algebra: transverse structure and commutant

3.1. Algebraic study: lifting homomorphism from $S(\mathfrak{g})/I$ into $\varprojlim_n S(\mathfrak{g})/I^n$: associated commutant

3.1.1. Invariant ideals, Poisson ideals and centralizing sequence of the symmetric algebra of a nilpotent Lie algebra

Let $\mathfrak{g}$ be a nilpotent finite-dimensional Lie algebra. Let us recall that the adjoint group of $\mathfrak{g}$ is the subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by the $\exp \mathrm{ad}\, x$, $x \in \mathfrak{g}$. We denote it by Γ. This group acts on $\mathfrak{g}$ and $S(\mathfrak{g})$ by automorphisms; It acts on $\mathfrak{g}^*$ by the contragradient operator. An ideal I of $S(\mathfrak{g})$ will be said Γ-invariant if $\Gamma \cdot I \subset I$. The Poisson ideals of $S(\mathfrak{g})$ are the Γ-invariant ideals of $S(\mathfrak{g})$. We shall use following Dixmier's proposition [8, Paragraph 4.2.2.5, p. 155]: Let $\mathfrak{g}$ be a nilpotent Lie algebra, and I a Poisson ideal of $S(\mathfrak{g})$. Any non-null Poisson ideal K of $S(\mathfrak{g})/I$ satisfies $K \cap C(S(\mathfrak{g})/I) \neq 0$.

We shall prove that any Poisson ideal of $S(\mathfrak{g})$ is generated by a centralizing sequence.

Theorem 28. *Let A be a Poisson algebra. Let us suppose that A is noetherian and satisfies the property:*

$(*)$ for all Poisson ideal J of A, any non-null Poisson ideal of A/J has a non-null intersection with the center $C(A/J)$ of A/J.

Then any Poisson ideal of A is generated by a centralizing sequence $(x_1, \ldots, x_n)$.

Proof. Let I be a Poisson ideal of A;

If $I = 0$, I is generated by a centralizing sequence; if $I \neq 0$, by the property $(*)$ applied with $J = 0$, we have $I \cap C(A) \neq 0$. Hence there exists a non-null central element of I, x_1; Ax_1 is a Poisson ideal.

If $I = Ax_1$, it is proved; if $Ax_1 \subset I$, let us set $A_1 = A/Ax_1$ and $I_1 = I/Ax_1$. The ideal I_1 is a non-null Poisson ideal of A/Ax_1. By virtue of the property $(*)$ applied to $J = Ax_1$ we have $I/Ax_1 \cap C(A/Ax_1) \neq 0$; therefore there exists $x_2 \in I$, $x_2 \neq x_1$ such that $\{a, x_2\} \in Ax_1$, for all $a \in A$. The sequence (x_2, x_1) is centralizing in A. If $I = Ax_2 + Ax_1$, it is proved; or else $Ax_2 + Ax_1 \subset I$.

Let us assume that in this way we have found a sequence $(x_n)_{n\in\mathbb{N}}$ such that $(x_1, \ldots, x_i)$ is centralizing for all $i \in \mathbb{N}$ and that again we have $Ax_i + Ax_{i-1} + \cdots + Ax_1 \neq I$ $\forall i \in \mathbb{N}$, then we have a strictly increasing sequence $Ax_1 \subset Ax_1 + Ax_2 \subset \cdots \subset Ax_h + Ax_{h-1} +$

$\cdots + Ax_1 \subset \cdots$ of ideals in A. But it is impossible, while we have assumed that the algebra A is noetherian; Therefore $\exists p \in \mathbb{N}$ *such that* $I = Ax_p + \cdots + Ax_1$. □

Corollary 29. *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra. Then any Poisson ideal* I *of* $S(\mathfrak{g})$ *is generated by a centralizing sequence.*

3.1.2. Lifting homomorphism associated to I rational ideal

Following Dixmier [8, paragraph 4.2.6], a Poisson ideal I of the symmetric algebra $S(\mathfrak{g})$ of a nilpotent Lie algebra $\mathfrak{g}$ is called a rational ideal if $C(S(\mathfrak{g})/I) = k$. According to Vergne [35] and Arnal et al. [1] or by introduction of Poisson algebra isomorphisms (and not only commutative algebras) in Nouazé-Gabriel's results [23, paragraphs 3.2 et 3.3], the fact, that to be rational for an ideal is equivalent to the existence of $p \in \mathbb{N}$ such that the Poisson algebra $S(\mathfrak{g})/I$ is isomorphic to $W_p(k)$.

Theorem 30. *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra,* I *a rational Poisson ideal of* $S(\mathfrak{g})$ *and* $\widehat{S}(\mathfrak{g}) = \underleftarrow{\lim}_n S(\mathfrak{g})/I^n$. *Then there exists a lifting homomorphism of Poisson algebras* φ : $S(\mathfrak{g})/I \longrightarrow \widehat{S}(\mathfrak{g})$. *The lifting homomorphism* φ *is unique up to inner Poisson isomorphism.*

Proof. We apply the lifting theorem Theorem 17 to $A = S(\mathfrak{g})$. From Corollary 29, any Poisson ideal of $S(\mathfrak{g})$ is generated by a centralizing sequence. The uniqueness of the lifting homomorphism φ is asserted by Theorem 19. □

3.1.3. Application to lifting homomorphism from $S(\mathfrak{g})/I(\mu)$ *into* $\underleftarrow{\lim}_n S(\mathfrak{g})/I^n$ *for* $I(\mu)$ *associated ideal to* μ *of* $\mathfrak{g}^*$*: regular centralizing sequence generating* $I(\mu)$

For $\mu \in \mathfrak{g}^*$, we call the set of polynomial functions on $\mathfrak{g}^*$ vanishing on the orbit $\Gamma \cdot \mu$ of μ the associated ideal to μ. We denote it $I(\mu)$. We know [8, Paragraph 6.3] that $I(\mu)$ is invariant rational, $S(\mathfrak{g})/I(\mu) \simeq W_p(k)$ for some p. Theorem 30 is applied to it.

Pukanszky's theorem [28] gives a parametrization of the coadjoint orbit through μ. From this parametrization, we show that it is easily possible to obtain generators $P_1, P_2, \ldots, P_n$ of the ideal $I(\mu)$. The sequence $(P_1, P_2, \ldots, P_n)$ is a regular centralizing sequence of $S(\mathfrak{g})$.

Let us recall Pukanszky's notations (see [28,29]) concerning the nilpotent Lie algebras (see [27, p. 426]). Let $\mathfrak{g}_0 = \{0\} \subset \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_m = \mathfrak{g}$ be a flag of $\mathfrak{g}$ ($\dim \mathfrak{g}_i = i$) such that $[\mathfrak{g}, \mathfrak{g}_i] \subseteq \mathfrak{g}_{i-1}$ for all $i \in \{1, \ldots, m\}$ and let $(X_1, \ldots, X_m)$ be an adapted basis called Jordan-Hölder, that is that $\mathfrak{g}_i = kX_1 \oplus \cdots \oplus kX_i$, for all $i \in \{1, \ldots, m\}$. The dual space $\mathfrak{g}^*$ of the algebra $\mathfrak{g}$ is provided with the following stratification:

For $\mu \in \mathfrak{g}^*$ we defined the set of indices $J_\mu = \{1 \leq j \leq m : X_j \notin \mathfrak{g}_{j-1} + \mathfrak{g}_\mu\}$, where $\mathfrak{g}_\mu = \{x \in \mathfrak{g} : \forall y \in \mathfrak{g}, \mu([x, y]) = 0\}$. If $J_\mu = \{j_1 < j_2 < \cdots < j_d\}$, we shall have $\mathfrak{g} = \mathfrak{g}_\mu \oplus kX_{j_1} \oplus \cdots \oplus kX_{j_d}$. Let $\Delta = \{J_\mu; \mu \in \mathfrak{g}^*\}$, for $e \in \Delta$, we define the subset of $\mathfrak{g}^*$

$$\Omega_e = \{\mu \in \mathfrak{g}^*; J_\mu = e\}$$

called stratum. We have $\mathfrak{g}^* = \cup_{e \in \Delta} \Omega_e$, disjoint finite union of strata.

Theorem 31 ([28] Parametrization of the orbit of $\mathfrak{g}^*$). *Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra m and Γ the adjoint group of $\mathfrak{g}$. Let $(X_1, \dots, X_m)$ be a Jordan-Hölder basis of $\mathfrak{g}$. Let $e = \{j_1 < j_2 < \cdots < j_d\}$ and Ω_e the corresponding stratum.*

Then, for all $j \in \{1, \dots, m\}$, there exist functions

$$\begin{aligned} R_j^e : \Omega_e \times k^d &\longrightarrow k \\ (\mu, y_{j_1}, \dots, y_{j_d}) &\longmapsto R^e(\mu, y_{j_1}, \dots, y_{j_d}) \end{aligned}$$

such that

(a) for $\mu \in \Omega_e$ fixed, the function

$$\begin{aligned} k^d &\longrightarrow k \\ (y_{j_1}, \dots, y_{j_d}) &\longmapsto R_j^e(\mu, y_{j_1}, \dots, y_{j_d}) \end{aligned}$$

is a polynomial function of the variables $y_{j_1}, \dots, y_{j_h}$, where h satisfies $j_h \leq j < j_{h+1}$;

(b) if $j = j_h \in e$, $R_{j_h}^e(\mu, y_{j_1}, \dots, y_{j_d}) = y_{j_h}$ $\forall \mu \in \Omega_e$;

(c) $\forall \mu \in \Omega_e$, the coadjoint orbit $\Gamma \cdot \mu$ through μ in $\mathfrak{g}^*$ is

$$\Gamma \cdot \mu = \left\{ \sum_{j=1}^{j=m} R_j^e(\mu, y_{j_1}, \dots, y_{j_d}) X_j^* \quad (y_{j_1}, \dots, y_{j_d}) \in k^d \right\},$$

where $(X_j^*)_{j \in \{1, \dots, m\}}$ is a dual basis $(X_j)_{j \in \{1, \dots, m\}}$.

For $\mu \in \mathfrak{g}^*$, we give generators of the ideal $I(\mu)$ of $S(\mathfrak{g})$ associated to μ.

Proposition 32. *With the hypotheses of Theorem 31, let μ in $\mathfrak{g}^*$. Then*

1. *$I(\mu) = \sum_{j \notin e}(X_j - R_j^e(\mu, X_{j_1}, \dots, X_{j_d}))S(\mathfrak{g})$;*
2. *the sequence $(X_j - R_j^e)_{j \notin e}$ is centralizing in $S(\mathfrak{g})$.*

Remark 33. *For $j \in e$, $k \in \{1, \dots, d\}$ $j = j_k$, we have the equality $X_{j_k} - R_{j_k}^e(\mu, X_{j_1}, \dots, X_{j_d}) = 0$ by virtue of Theorem 31. We see, from the form of the generators $P_j = X_j - R_j^e(\mu, X_{j_1}, \dots, X_{j_d})$, that the sequence $(P_j)_{j \notin e}$ generating the ideal $I(\mu)$ is regular. For $\nu \in \Omega = G \cdot \mu$ and $P_j = X_j - R_j^e$, $j \notin e$, we have*

$$\mathrm{d}P_j(\nu) \in \mathfrak{g}_\nu, \quad \mathfrak{g}_\nu = \oplus_{j \notin e} \mathrm{d}P_j(\nu), \quad \mathfrak{g} = (\oplus_{j \in e} \mathbb{R} X_j) \oplus \mathfrak{g}_\nu$$

(see the following example with $\mathfrak{g} = \mathfrak{g}_{54}$).

Proof.

(1) Let $e = \{j_1 < \cdots < j_d\}$, Ω_e be the associated stratum and $\mu \in \Omega_e$. Let us prove $\sum_{j \notin e}(X_j - R_j^e(\mu, X_{j_1}, \dots, X_{j_d}))S(g) \subset I(\mu)$. It is enough to check that $\varphi(X_j - R_j^e(\mu, X_{j_1}, \dots, X_{j_d})) = 0, \forall \varphi \in G \cdot \mu, \forall j \notin e$, that is to say $\varphi_j - R_j^e(\mu, \varphi_{j_1}, \dots, \varphi_{j_d}) = 0$ denoting $\varphi(X_j) = \varphi_j$. The element $\varphi \in \Gamma \cdot \mu$ implies $\varphi = \sum_{j=1}^{j=m} R_j^e(\mu, y_{j_1}, \dots,$

$y_{j_d})X_j^*$ for some $y \in k^d$, but $\varphi_{j_r} = R_{j_r}^e(\mu, y) = y_{j_r}$ from the definition of R_j^e (voir recall), $r \in \{1, \ldots, d\}$ and therefore $\varphi_j = R_j^e(\mu, \varphi_{j_r}, \ldots, \varphi_{j_d})$.

It remains to show that

$$I(\mu) \subset \sum_{j \notin e} (X_j - R_j^e(\mu, X_{j_1}, \ldots, X_{j_d}))S(g).$$

Let $v \in I(\mu)$. Let us denote $\bar{e} = \{i_1 < i_2 < \cdots < i_k\} = \{1, \ldots, m\} - e$, $v(X_{i_1}, \ldots, X_{i_k}, X_{j_1}, \ldots, X_{j_d}) = \widetilde{v}(X_{i_1}, \ldots, X_{i_k})$ considering v as function of the only variables X_{i_l}. We apply Taylor's formula to $\widetilde{v}$ at the $(R_{i_1}(\mu, X_{j_1}, \ldots, X_{j_d}), \ldots, R_{i_k}(\mu, X_{j_1}, \ldots, X_{j_d}))$.

(2) Let us set $P_j = X_j - R_j$ with $j \notin e$ ($P_j = 0$ for $j \in e$). We notice that the elements $P_1, \ldots, P_j$ form a basis of $I \cap S(\mathfrak{g}_j)$ and give the equations of the projection of the orbit $\Omega = G.\mu$ on $\mathfrak{g}^*{}_j$. Indeed the element R_j^e depends only on the variables X_l with $l \leq j$, $l \in e$. We have $I \cap S(\mathfrak{g}_j) = \sum_{j' \leq j} S(\mathfrak{g}_j)P_{j'}$. For $X \in \mathfrak{g}$, the bracket $\{P_j, X\} = \{X_j, X\} - \{R_j^e, X\}$ only depends on the variables $X_{j'}$ with $j' < j$ while we have $\{\mathfrak{g}, \mathfrak{g}_j\} \subset \mathfrak{g}_{j-1}$ and $R_j^e \in S(\mathfrak{g}_j)$. Thus we obtain

$$\{P_j, X\} \in S(\mathfrak{g}_{j-1}) \cap I = \sum_{j' < j} S(\mathfrak{g}_{j-1})P_{j'}$$

the sequence $P_1, \ldots, P_j$ is a centralizing sequence of $S(\mathfrak{g})$. □

Example 34. Let $\mathfrak{g} = \mathfrak{g}_{41}$ be the nilpotent Lie algebra (see [26, p. 12]) having for basis $(X_1, \ldots, X_4)$ and which the Lie algebra structure is defined by the brackets: $[X_4, X_3] = X_2$ et $[X_4, X_2] = X_1$. Let $e = \{j_1 < j_2\} = \{2, 4\}$ and $\Omega_e = \{\sum_{i=1}^{i=4} \xi_i X_i^*; \xi_1 \neq 0\}$ the corresponding stratum. Let $\mu = \sum_{i=1}^{i=4} \mu_i X_i^* \in \Omega_e$, we have

$$I(\mu) = (X_1 - \mu_1)S(\mathfrak{g}_{41}) + (2\mu_1 X_3 - X_2^2 + \mu_2^2 - 2\mu_1\mu_3)S(\mathfrak{g}_{41}).$$

3.1.4. Commutant associated to a lifting homomorphism

Theorem 35. *Let $\mathfrak{g}$ be a nilpotent Lie algebra, $\mu \in \mathfrak{g}^*$ and $I(\mu)$ the Poisson ideal associated to μ. Let us denote $\varphi : S(\mathfrak{g})/I(\mu) \longrightarrow \widehat{S}$ a lifting homomorphism (Theorem 17). Let C be the commutant of $\varphi(S(\mathfrak{g})/I(\mu))$ in $\widehat{S}$. Let $m = \widehat{I} \cap C$. Then*

(i) *The Poisson algebra C is a local ring with residue field k, separated completely for the m-adic topology, m being the maximal ideal*

(ii) *C is isomorphic as an associative commutative unitary algebra to the formal power series algebra $k[[X_1, \ldots, X_r]]$, the integer r is the length of a regular centralizing sequence generating $I(\mu)$, it is as well the codimension of the orbit through μ.*

Proof. We apply Theorem 23 to $A = S(\mathfrak{g})$, $I(\mu)$ generated by a regular centralizing sequence from Proposition 32.

3.1.5. Examples of lifting homomorphism of $S(\mathfrak{g})/I$ into $\varprojlim_n S(\mathfrak{g})/I^n(\mu)$ and associated commutant

We show lifting examples.

In Example 36, we shall notice that the form of the formal lifting maps corresponds either to *rational functions* or to *algebraic functions*. In Example 37, we compare our results with those of Fokko du Cloux [11, pp. 198–199], and there is a strong similarity between the formulae giving the lifting homomorphism and the commutant in the respective spaces $\widehat{S(\mathfrak{g})}$ and $\widehat{U(\mathfrak{g})}$. Only the graded algebras of the commutants in their maximal ideals are given by different formulae.

Example 36. We consider the nilpotent Lie algebra $\mathfrak{g}_{54}$ [22,26], with basis $(X_1, \dots, X_5)$ and which the non-null brackets $[X_i, X_j]$, $i > j$, are given by: $[X_5, X_4] = X_3$; $[X_5, X_3] = X_2$; $[X_4, X_3] = X_1$. The dual basis of $\mathfrak{g}_{54}^*$ is $(X_1^*, X_2^*, \dots, X_5^*)$. We give a lifting homomorphism at a point ξ in every stratum denoted by $\Omega_1, \Omega_2, \Omega_3, \Omega_4$. Let $\pi : S(\mathfrak{g}) \longrightarrow S(\mathfrak{g})/I(\xi)$ be the canonical projection.

- We choose in $\mathfrak{g}_{54}^*$ at the point $\xi \in \Omega_1 = \{\xi = \sum \xi_i X_i^* : \xi_1 \neq 0\}$. The Poisson–Weyl algebra is $S(\mathfrak{g})/I(\xi) = W_1(k) = k[p, q]$ where $p = \pi(\xi_1^{-1} X_4)$ and $q = \pi(X_3)$. The lifting homomorphism of Poisson algebras, $\varphi : S(\mathfrak{g})/I(\xi) \longmapsto \widehat{S}$, is given by $\varphi(p) = X_4 X_1^{-1}$ and $\varphi(q) = X_3$, where X_1^{-1} is the element $\sum_{h=0}^{h=\infty} (-1)^h (X_1 - \xi_1)^h / (\xi_1)^{h+1}$ of $\widehat{S}$.

  This formal lifting map comes from rational functions, it is in accordance with Vergne's theorem [35, p. 327], *the stratum Ω_1 is the Zariski open set of this theorem, we have $\{X_4, X_3\} = X_1$, X_1 is the central element.*

  The commutant $C = C(W_1(\mathbb{R}), \widehat{S})$ has for maximal ideal m generated by u_1, u_2, u_3, where $u_1 = X_1 - \mu_1$, $u_2 = X_2$ and $u_3 = (2X_1X_5 + X_3^2 - 2X_2X_4)2^{-1}X_1^{-1}$ and X_1^{-1} is the element $\sum_{h=0}^{h=\infty} (-1)^h (X_1 - \xi_1)^h / (\xi_1)^{h+1}$ of $\widehat{S}$. The commutant is the algebra $C = k[[u_1, u_2, u_3]]$, trivial Poisson algebra.
- We choose in $\mathfrak{g}_{54}^*$ a point $\xi \in \Omega_2 = \{\xi = \sum \xi_i X_i^* : \xi_1 = 0, \xi_2 \neq 0\}$.

  The Poisson–Weyl algebra is $S(\mathfrak{g})/I(\xi) = W_1(k) = k[p, q]$ where $p = \pi(\xi_2^{-1} X_5)$ and $q = \pi(X_3) = X_3$. The lifting homomorphism of Poisson algebras $\varphi : S(\mathfrak{g})/I(\xi) \longmapsto \widehat{S}$, is given by $\varphi(p) = X_5 X_2^{-1}$ and $\varphi(q) = X_3$, where X_2^{-1} is the element $\sum_{h=0}^{h=\infty} (-1)^h (X_2 - \xi_2)^h / (\xi_2)^{h+1}$ of $\widehat{S}$. *Again this formal lifting map comes from rational functions.*

  The commutant $C = C(W_1(\mathbb{R}), \widehat{S}))$ has for maximal element m generated by u_1, u_2, u_3 where $u_1 = X_1$, $u_2 = X_2 - \mu_2$ and $u_3 = (X_3^2 + 2X_1X_5 - 2X_2X_4)2^{-1}X_2^{-1}$, X_2^{-1} is the element $\sum_{h=0}^{h=\infty} (-1)^h (X_2 - \xi_2)^h / (\xi_2)^{h+1}$ de $\widehat{S}$. The commutant is the algebra $C = k[[u_1, u_2, u_3]]$, trivial Poisson algebra.
- We choose in $\mathfrak{g}_{54}^*$ a point $\xi \in \Omega_3 = \{\xi = \sum \xi_i X_i^* : \xi_1 = 0, \xi_2 = 0, \xi_3 \neq 0\}$.

The Poisson–Weyl algebra is $S(\mathfrak{g})/I(\xi) = W_1(k) = k[p, q]$ where $p = \pi(\xi_3^{-1} X_5)$ and $q = \pi(X_4) = X_4$. The lifting homomorphism of Poisson algebras $\varphi : S(\mathfrak{g})/I(\xi) \longmapsto \widehat{S}$, is given by $\varphi(p) = 2X_5/(X_3 + \sqrt{X_3^2 + 2X_1X_5})$ et $\varphi(q) = X_4$. The element $\varphi(p)$ is

$$\varphi\left(\pi\left(\frac{X_5}{\xi_3}\right)\right) = X_5X_3^{-1} + \sum_{n=2}^{n=\infty}(-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2\cdot 4\cdot 6\cdots 2n}2^n X_5^n X_1^{n-1} X_3^{-2n+1}$$

of $\widehat{S}$, with X_3^{-1} the following element of $\widehat{S}$: $\sum_{h=0}^{h=\infty}(-1)^h(X_3-\xi_3)^h/(\xi_3)^{h+1}$.

This formal lifting comes from algebraic functions, it seems impossible to find a lifting homomorphism formed of rational functions.

The commutant $C = C(W_1(\mathbb{R}), \widehat{S})$ has for maximal ideal m generated by u_1, u_2, u_3, where $u_1 = X_1$, $u_2 = X_2$ et $u_3 = \sqrt{X_3^2 + 2X_1X_5} - \mu_3$, u_3 is the element

$$X_3 - \mu_3 + X_1X_5X_3^{-1} + \sum_{k=2}^{k=\infty}\frac{(-1)^{k+1}1\cdot 3\cdot 5\cdots(2k-3)}{2\cdot 4\cdot 6\cdots 2k}2^k X_5^k X_3^{-2k+1} X_1^k$$

in $\widehat{S}$. The commutant is the algebra $C = k[[u_1, u_2, u_3]]$, trivial Poisson algebra.

Let us recall Pedersen's theorem [25, p. 547] expressed in algebraic terms: Let Ω_e be a stratum of $\mathfrak{g}^*$ and $\xi \in \Omega_e$, Pedersen describes a homomorphism of associative algebras $\theta : S(\mathfrak{g})/I(\xi) \longrightarrow S(\mathfrak{g})_{I(\xi)}$, such that the following diagram be commutative:

$$\begin{tikzcd} S(\mathfrak{g})_{I(\xi)} \arrow[r, "p"] & Frac(S(\mathfrak{g})/I(\xi)) \\ & S(\mathfrak{g})/I(\xi) \arrow[u, "i"] \arrow[ul, "\theta"] \end{tikzcd}$$

where i is the canonical injective map of $S(\mathfrak{g})/I(\xi)$ in its fractions field and $S(\mathfrak{g})_{I(\xi)}$ is the localization of $S(\mathfrak{g})$ at $I(\xi)$. The map $p : S(\mathfrak{g})_{I(\xi)} \longrightarrow S(\mathfrak{g})_{I(\xi)}/I(\xi)S(\mathfrak{g})_{I(\xi)}$ is the canonical projection, and we use the isomorphism $S(\mathfrak{g})_{I(\xi)}/I(\xi)S(\mathfrak{g})_{I(\xi)} \simeq \mathrm{Frac}(S(\mathfrak{g})/I(\xi))$.

In the above example, Pedersen's calculus leads to a homomorphism of Poisson algebras on the strata Ω_1 and Ω_2. On the stratum Ω_3, Pedersen's homomorphism defined by $\theta(X_5\xi_3^{-1}) = X_5/X_3$ and $\theta(X_4) = X_4$ is not a Poisson homomorphism since we have $\{X_5/X_3, X_4\} = 1 + X_1X_3^{-2}X_5$.

Example 37 (*Comparison of lifting maps in $\widehat{S}$ and in $\hat{U}$*). Let us consider the nilpotent Lie algebra $\mathfrak{g}_{53}$ with basis $(X_1, \ldots, X_5)$ (see [26, p. 20]). The brackets of $\mathfrak{g}_{53}$ satisfy $[X_5, X_4] = X_2$, $[X_5, X_2] = X_1$, $[X_4, X_3] = X_1$. With the notations of Section 3.1.4, for $e = \{j_1 < j_2\} = \{4, 5\}$, the corresponding stratum is $\Omega_e = \{\xi = \sum_{i=1}^{i=5}\xi_i X_i^*; \xi_1 = 0, \xi_2 \neq 0\}$. Let us fix an element $\mu \in \Omega_e$. Then we have,

- The orbit of μ, $\Gamma\cdot\mu$: $\Gamma\cdot\mu = \{\mu_2X_2^* + \mu_3X_3^* + y_4X_4^* + y_5^*X_5(y_4, y_5) \in k^2\}$.
- The invariant rational ideal associated to μ : $I(\mu) = X_1S(\mathfrak{g}_{53}) + (X_2 - \mu_2)S(\mathfrak{g}_{53}) + (X_3 - \mu_3)S(\mathfrak{g}_{53})$.

- The rational ideal J of U associated to the orbit $\Gamma\cdot\mu$: $J = X_1U(\mathfrak{g}_{53})+(X_2-\mu_2)U(\mathfrak{g}_{53})+(X_3-\mu_3)U(\mathfrak{g}_{53})$.
- A lifting homomorphism $\varphi : S(\mathfrak{g}_{53})/I(\mu)\longrightarrow\widehat{S} = \varprojlim_n S(\mathfrak{g}_{53})/I^n(\mu)$ given by $\varphi(\overline{X_5}/\mu_2) = X_5X_2^{-1}$ and $\varphi(\overline{X_4}) = X_4$ with X_2^{-1} the element $X_2^{-1} = \sum_{k=0}^{k=\infty}(-1)^k(X_2-\mu_2)^k/\mu_2^{k+1}$.
- A lifting homomorphism $\phi : U(\mathfrak{g}_{53})/J\longrightarrow\widehat{U} = \varprojlim_n U(\mathfrak{g}_{53})/J^n$ given by du Cloux and defined by $\phi(\overline{X_5}/\mu_2) = X_5X_2^{-1}$and $\phi(\overline{X_4}) = X_4$.
- The commutant $C = C(W_1(\mathbb{R}), \widehat{S})$ with maximal ideal m generated by u_1, u_2, u_3:
  $m = (u_1, u_2, u_3)_{\widehat{S}}$, where

$$u_1 = X_1, u_2 = \sqrt{X_2^2-2X_1X_4}-\mu_2 \quad\text{and}\quad u_3 = X_1X_5X_2^{-1}+X_3,$$

  where u_2 is the element $X_2-\mu_2-X_1X_4X_2^{-1}-\sum_{k=2}^{k=\infty}(1\,3\,5\,\cdots\,(2k-3)/2\,4\,6\,\cdots\,2k)\,2^kX_4^kX_2^{-2k+1}X_1^k$ de $\widehat{S}$.
- The commutant $D = C(A_1(\mathbb{R}), \widehat{U}))$ with maximal ideal n generated by v_1, v_2, v_3:
  $n = (v_1, v_2, v_3)_{\widehat{U}}$ where

$$u_1 = X_1, v_2 = \sqrt{X_2^2-2X_1X_4}-\mu_2 \quad\text{and}\quad v_3 = X_1X_5X_2^{-1}+X_3,$$

  where v_2 is the element $X_2-\mu_2-X_1X_4X_2^{-1}-\sum_{k=2}^{k=\infty}(1\,3\,5\,\cdots\,(2k-3)/2\,4\,6\,\cdots\,2k)2^kX_4^kX_2^{-2k+1}X_1^k$ de $\widehat{U}$.
- The commutant in $\widehat{S}$ is the formal power series $C = k[[u_1, u_2, u_3]]$, Poisson algebra with the law $\{u_3, u_2\} = u_1^2(u_2+\mu_2)^{-1}$.
- The commutant in $\widehat{U}$ is the algebra of non-commutative formal power series $D = k[[v_1, v_2, v_3]]$, which law algebra is given by $[v_3, v_2] = v_1^2(v_2+\mu_2)^{-1}$.

  The respective commutant formulae are identical, this leads to conjecture that the associative algebra D is a quantization of the Poisson algebra C.
- The m-adic graded algebra of C is the polynomial algebra $\mathrm{Gr}_m C = k[t_1, t_2, t_3]$ endowed with the trivial Poisson.
- The n-adic graded algebra of D calculated by du Cloux is the algebra $\mathrm{Gr}_n D = k[t_1, t_2, t_3]$ described by $[t_3, t_2] = t_3^2/\mu_2$.

  The formulae of the respective graded algebras differ.

3.2. Geometric study: commutant and transverse structure

Let $\mathfrak{g}$ be a real finite-dimensional nilpotent algebra. The dual space $\mathfrak{g}^*$ is a Poisson manifold endowed with its Lie–Poisson structure. The goal of this section is to compare the formal Poisson algebra transverse to a symplectic leaf, algebra obtained by Taylor series expansion of the transverse algebra of Weinstein's splitting theorem, with the commutant $\varphi S(\mathfrak{g})/I(\mu))$ in $\widehat{S}$, where φ is a lifting homomorphism of $S(\mathfrak{g})$ into $\widehat{S}$, and μ is a point of the considered leaf.

3.2.1. Transverse Poisson structure

The structure of Poisson manifold has been defined and studied by Lichnerowicz [19]. The local structure of a Poisson manifold has been specified by Weinstein [36] in Weinstein's splitting theorem. Let us recall the definition:

Let M be a smooth manifold This manifold M will be called Poisson manifold if the algebra of infinitely differentiable functions $C^{\infty}(M)$ is endowed with a Poisson algebra structure. There exists on M a unique two times contravariant skew-symmetric tensor field, infinitely differentiable denoted by Λ such that for f and g in $C^{\infty}(M)$: $\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g)$. The tensor field Λ is called the Poisson tensor field of the Poisson manifold M. A Poisson manifold will be denoted (M, Λ). The map $\Lambda^{\sharp} : T^{*}M \longrightarrow TM$ such that for all x in M, α, β in $T_x^{*}M$ we have $\langle \Lambda^{\sharp}_{(x)}(\alpha), \beta\rangle = \Lambda_{(x)}(\alpha, \beta)$, is a morphism of the cotangent bundle into the tangent bundle, $\langle \cdot, \cdot \rangle$ being the duality between $T_x^{*}M$ and T_xM.

Let M be a Poisson manifold. Let N be an immersed submanifold of M. Let us assume that N is endowed with a Poisson manifold structure. The submanifold N will be called Poisson submanifold of the Poisson manifold M if the injection $i : N \longrightarrow M$ is a Poisson morphism.

Weinstein has proved [36] the following:

Theorem 38. *For $x \in N$, let $T_xN^{\perp} = \{\varphi \in T_x^{\star}M \ : \forall u \in T_xN \ \varphi(u) = 0\}$. We assume that:* (i) $\forall x \in N$, $\Lambda^{\sharp}_M(x)(T_xN^{\perp}) \cap T_xN = \{0\}$; (ii) $\forall x \in N$, $T_xN^{\perp} \cap \operatorname{Ker} \Lambda^{\sharp}_M(x) = \{0\}$. *Then N is endowed with the Poisson structure (N, Λ_N) such that the vector bundles morphism $\Lambda^{\sharp}_N : T^{\star}N \longrightarrow TN$ is*

$$\Lambda^{\sharp}_N = \pi \circ \Lambda^{\sharp}_M \circ \pi^{\star},$$

where for x in N, the map $\pi_x : T_xM \longrightarrow T_xN$ is the map associated to the decomposition $T_xM = T_xN \oplus \Lambda^{\sharp}_M(x)(T_xN^{\perp})$. This structure is called the Poisson structure on N induced by M. The induced Poisson bracket is for $f, g \in C^{\infty}(N)$: $\{f, g\}_N = \Lambda_M(\mathrm{d}f \circ \pi, \mathrm{d}g \circ \pi)$.

Let (M, Λ) be a Poisson manifold, the set $D = \Lambda^{\sharp}(T^{\star}M)$, image of the cotangent bundle by the bundle morphism $\Lambda^{\sharp}$, defines a C^{∞} distribution on the manifold M in Sussmann's terminology [32]. For all point y in M, there exists a unique immersed connected submanifold S of M, maximal for inclusion, y in S, such that for all x of S we have $T_xS = \Lambda^{\sharp}_x(T_x^{\star}M)$. The manifold S is called the symplectic leaf through y in M. It is obvious that every leaf S is a Poisson submanifold.

Let us recall the definition of the transverse structure to a symplectic leaf for a Poisson manifold. Let (M, Λ) be a Poisson manifold. Let S be a symplectic leaf M, $x_0 \in S$ and N a submanifold of M, with dimension the codimension of S in M, going through x_0 and transverse to S at x_0 : $T_{x_0}M = T_{x_0}S \oplus T_{x_0}N$. There exists a neighborhood U of x_0 in N such that, for all x in U, we have $T_xM = \Lambda^{\sharp}((T_xN)^{\perp}) + T_xN$. By virtue of Theorem 38 and its remark, we can endow $U \subset N$ with the Poisson structure induced by M. It has been proved by Weinstein [36] (see also [6, Theorem 6.2]):

Let x_0 and x_1 be two points of the symplectic leaf S and two submanifolds N_0 and N_1 such that $x_0 \in N_0$, $x_1 \in N_1$, and

$$T_{x_i}M = T_{x_i}S \oplus T_{x_i}N_i, \quad i \in \{1, 2\},$$

we provide N_0 et N_1 with the Poisson structures induced by M, then there exists an isomorphism of Poisson manifolds from a neighborhood of x_0 in N_0 onto a neighborhood of x_1 in N_1 which maps x_0 on x_1. The notion of Poisson manifolds isomorphism defines an equivalence relation on the set of Poisson manifolds. We have the following definition.

Definition 39. Let M be a Poisson manifold, S a symplectic leaf and $x_0 \in S$. Let N be a submanifold of M through x_0 and transverse to S at x_0, the equivalence class of the germ at x_0 of the Poisson structure induced on N by M will be called transverse Poisson structure to the leaf S.

Let us recall Dirac's constraints formula. Let (M, Λ) be a Poisson manifold and N an immersed submanifold satisfying the conditions of Theorem 38, then N is provided with the Poisson structure induced by that of M. The formula by Dirac [7] has given a relationship between the bracket on N and the bracket on M.

Theorem 40 (Dirac's bracket formula). *Let M be a Poisson manifold. Let N be a submanifold satisfying the conditions* (i) *et* (ii) *of Theorem 38. Let x_0 be in M. We choose U open set of M, $x_0 \in U$ and functions x^α such that we have*

$$N \cap U = \{y \in U : x^1(y) = x^2(y) = \cdots = x^{2k}(y) = 0\},$$

such that the matrix

$$[\{x^\alpha, x^\beta\}(y)], \quad \alpha, \beta \in \{1, \ldots, 2k\}$$

be invertible for all $y \in N \cap U$. We denote for $y \in N \cap U$, $\alpha, \beta \in \{1, \ldots, 2k\}$ $C^{\alpha\beta}(y) = \{x^\alpha, x^\beta\}(y)$; Let $C_{\mu\nu}$ be such that for $\alpha, \gamma \in \{1, \ldots, 2k\}$ $\sum_{\beta=1}^{\beta=2k} C_{\alpha\beta}(y)C^{\beta\gamma}(y) = \delta_\alpha^\gamma$. Let us denote $i : N \longrightarrow M$ the canonical injection.

Then the relation between the Poisson structure induced on N and that of M is given by

$$\{f \circ i, g \circ i\}_N = \{f, g\}_M - \sum_{\alpha.\beta=1}^{\alpha.\beta=2k} \{f, x^\alpha\}_M(y) C_{\alpha\beta}(y) \{x^\beta, g\}_M(y)$$

for all $f, g \in C^\infty(M)$ $\alpha, \gamma \in \{1, \ldots, 2k\}$.

Proof. See [2,31]. □

3.2.2. Comparison of the commutant with the formal transverse Poisson algebra deduced from Weinstein's theorem

We recall Weinstein's fundamental splitting theorem [36] given in its version C^∞. It allows us to calculate the transverse structure to a symplectic leaf of a Poisson manifold (see Definition 39).

Theorem 41 (Weinstein's theorem (version C^∞)). *Let M be a Poisson manifold,* $\dim M = n$. *Let $x_0 \in M$. We assume that the rank of Λ at x_0 is $2r$.*

Then,

1. *There exists a chart $(U, \phi, \mathbb{R}^n)$ of M at x_0, called a Weinstein chart at x_0, such that the associated coordinate maps $(p_1, \ldots, p_r, q_1, \ldots, q_r, z_1, \ldots, z_{n-2r})$, for $i, j \in \{1, \ldots, r\}$, $\alpha, \beta \in \{1, \ldots, n-2r\}$ satisfy:*

$$(*) \qquad \begin{cases} \{p_i, p_j\} = 0 & \{q_i, q_j\} = 0 \quad \{p_i, q_j\} = \delta_{ij} \text{ (Kroenecker delta)}, \\ \{p_i, z_\alpha\} = 0 & \{q_i, z_\alpha\} = 0, \\ \{z_\alpha, z_\beta\}(x_0) = 0. \end{cases}$$

   The brackets $\{z_\alpha, z_\beta\}$ are only dependent on the $z_1, \ldots, z_{n-2r}$.
2. *There exists an open set U including x_0 which is identified by a Poisson isomorphism to a product $V \times W$, V open set of $\mathbb{R}^{2r}$ endowed with the canonical symplectic structure, W open set of $\mathbb{R}^{n-2r}$ endowed with a structure of Poisson manifold which the rank of the associated tensor is null at the projection point of x_0 on W. The factors V and W are unique up to local Poisson isomorphism.*

If N is the manifold transverse to the leaf through x_0, we have with $\varphi_{(x_0)} = (p_0, q_0, z_0)$

$$N \cap U = \{x \in U : p = p_0, q = q_0\}$$

for all $\phi, \psi \in C^\infty(N)$.

The Poisson algebra transverse to the leaf through x_0, $C^\infty(N)$, is provided with the bracket

$$\{\phi, \psi\}(z) = \sum_{1 \le \alpha, \beta \le n-2r} \{z_\alpha, z_\beta\} \frac{\partial \phi}{\partial z_\alpha} \frac{\partial \psi}{\partial z_\beta}(z).$$

The algebra $C^\infty(N)$ is isomorphic to Weinstein's commutant of the p and q in $C^\infty(M)$. For examples in dimension six, we must consult [31, pp. 91–92].

Example 42. We still consider the nilpotent Lie algebra $\mathfrak{g} = \mathfrak{g}_{5,3}$ [22,26].

We choose linear coordinates, $\sigma : \mathfrak{g}^* \longrightarrow \mathbb{R}^5$ which to $\sum_{i=1}^{i=5} x_i X_i^*$, associate $(x_i)_{i \in \{1,\ldots,5\}}$. In these coordinates, the Poisson bracket is for all $f, g \in C^\infty(\mathbb{R}^5)$ $\{f, g\} = -x_2 \Lambda_{4,5}(f, g) - x_1 \Lambda_{2,5}(f, g) - x_1 \Lambda_{3,4}(f, g)$, with $\Lambda_{i,j}(f, g) = (\partial f/\partial x_i)(\partial g/\partial x_j) - (\partial f/\partial x_j)(\partial g/\partial x_i)$.

First case. $\mu = \sum_{i=1}^{i=5} \mu_i X_i^*$ avec $\mu_1 \neq 0$. A Weinstein chart, defined on an open set U including μ, is $(p_1, q_1, p_2, q_2, z_1)$:

$$p_1 = \frac{x_1(x_5 - \mu_5) + (x_2 - \mu_2)(x_3 - \mu_3)}{x_1^2}, \quad q_1 = x_2 - \mu_2, \quad p_2 = \frac{x_4 - \mu_4}{x_1},$$
$$q_2 = x_3 - \mu_3, \quad z_1 = x_1.$$

Second case. $\mu = \sum_{i=1}^{i=5} \mu_i X_i^*$ avec $\mu_1 = 0$ et $\mu_2 \neq 0$. A Weinstein chart $(p_1, q_1, z_1, z_2, z_3)$ defined on an open set U including μ, is

$$p_1 = x_4 - \mu_4, \quad q_1 = -\frac{(x_5 - \mu_5)}{x_2}, \quad z_1 = x_1,$$

$$z_2 = \sqrt{x_2^2 - 2x_1(x_4 - \mu_4)} - \mu_2, \quad z_3 = x_1 \frac{(x_5 - \mu_5)}{x_2} + x_3 - \mu_3.$$

Computation of the brackets $\{z_i, z_j\}_{i,j \in \{1,\dots,3\}}$: the element z_1 being central, we must only calculate $\{z_2, z_3\}$:

$$\{z_2, z_3\} = -\frac{x_1^2}{\sqrt{x_2^2 - 2x_1(x_4 - \mu_4)}}, \quad \{z_2, z_3\} = -\frac{z_1^2}{z_2 + \mu_2}.$$

The submanifold N transverse to the leaf through μ is: $N \cap U = \{x \in \mathfrak{g}^* : p_1(x) = 0, q_1(x) = 0\}$.

In the coordinates (z_1, z_2, z_3) we have: $\varphi, \phi \in C^\infty(N)$ $\{\varphi, \phi\}_{(z_1, z_2, z_3)} = -(z_1^2/z_2 + \mu_2)\Lambda_{2,3}(\varphi, \phi)$.

Theorem 43 (Weinstein's formal theorem). *Let F be an algebra of formal power series over k in n indeterminates and $\mathfrak{m}$ the maximal ideal of F. Let us assume that F is a Poisson algebra. Then,*

1. *There exists an n-coordinate system $(p_1, \dots, p_r, q_1, \dots, q_r z_1, \dots, z_{n-2r})$ formed of elements of $\mathfrak{m}$, called a Weinstein formal system with $i, j \in \{1, \dots, r\}, \alpha, \beta \in \{1, \dots, n-2r\}$, such that:*

$$\begin{cases} F \simeq k[[p_1, \dots, p_r, q_1, \dots, q_r, z_1, \dots, z_{n-2r}]] \quad \text{where} \\ \{p_i, p_j\} = 0 \quad \{q_i, q_j\} = 0 \quad \{p_i, q_j\} = \delta_{ij} \\ \{p_i, z_\alpha\} = 0 \quad \{q_i, z_\alpha\} = 0 \\ \{z_\alpha, z_\beta\} \in \mathfrak{m}. \end{cases}$$

   The brackets $\{z_\alpha, z_\beta\}$ belong to $k[[z]]$.

2. *The formal transverse Poisson algebra $k[[z]]$ endowed with the brackets*

$$\{u, v\} = \sum_{1 \le \alpha, \beta \le n-2r} \{z_\alpha, z_\beta\} \frac{\partial u}{\partial z_\alpha} \frac{\partial v}{\partial z_\beta}, \quad u, v \in k[[z]]$$

   is unique up to isomorphism of Poisson algebras.

Proof. We repeat mutatis mutandis the proof of thes C^∞ version of the theorem. To show the uniqueness, in the case where $k = \mathbb{R}$ which will interest us further, we can use also Borel's theorem [34] which asserts that the map

$$\pi : C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}[[X_1, \dots, X_n]]$$

associating to every function its Taylor's series is surjective. Then we use Weinstein's previous theorem in version C^∞. □

Weinstein's theorem has allowed us to calculate the transverse Poisson algebra at a point μ of $\mathfrak{g}^*$, algebra isomorphic to the Weinstein commutant at μ. We expect the algebra obtained by Taylor's series expansion of Weinstein's commutant to be isomorphic to our commutant computed in $\widehat{S(\mathfrak{g})}$. In fact, we have computed the commutant in $\widehat{S}$ by successive approximations, The Weinstein commutant is worked out by resolution of differential equations. To compare the commutant obtained in $\widehat{S(\mathfrak{g})}$ and the algebra obtained by expansion of the Weinstein commutant in formal power series, we will immerse these two algebras into a formal power series algebra.

Still we use the Pukanszky and Pedersen's notations (see Section 3.1.3). Let $\{0\} = \mathfrak{g}_0 \subseteq \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_m = \mathfrak{g}$ be a flag of the Lie $\mathbb{R}$-algebra.

$\mathfrak{g}$ satisfying $[\mathfrak{g}, \mathfrak{g}_i] \subseteq \mathfrak{g}_{i-1}$ for all $i \in \{1, \dots, m\}$ and let $(X_1, \dots, X_m)$ be an adapted basis to the flag, $\mathfrak{g}_i = \mathbb{R}X_1 \oplus \cdots \oplus \mathbb{R}X_i$. Let Ω be an orbit of $\mathfrak{g}^*$ and $\mu \in \Omega$, then μ belongs to a stratum Ω_e for some $e = \{j_1 < \cdots < j_d\}$ with $d = \dim \Omega$. Let us set $\bar{e} = \{1, \dots, m\} - e = \{i_1 < \cdots < i_h\}$. We know that with the notations and results of Proposition 32, the associated ideal to μ is

$$I(\mu) = \sum_{l=1}^{l=h} (X_{i_l} - R^{\mathrm{e}}_{i_l}(\mu, X_{j_1}, X_{j_2}, \dots, X_{j_\alpha}))S(\mathfrak{g}),$$

with $j_\alpha \le i_l < j_{\alpha+1}$. Let us set $T_{j_k} = X_{j_k} - \mu_{j_k}$, $k \in \{1, \dots, k\}$ et $U_{i_l} = X_{i_l} - R^e_{i_l}(\mu, X_{j_1}, X_{j_2}, \dots, X_{j_\alpha})$, $l \in \{1, \dots, h\}$. Then we have

$$\widehat{S} = \mathbb{R}[T_{j_1}, \dots, T_{j_d}][[U_{i_1}, \dots, U_{i_h}]].$$

We notice that an element of $\widehat{S(\mathfrak{g})}$ is a formal power series with polynomial coefficients of $R[T]$.

Let us consider the Poisson manifold $(\mathfrak{g}^*, \Lambda)$, the rank of Λ is d at μ.

There exists a chart

$$\begin{aligned} \varphi : U &\longrightarrow \mathbb{R}^m \quad \text{at } \mu, \\ x &\longmapsto (p_1, \dots, p_r, q_1, \dots, q_r, z_1, \dots, z_h), \end{aligned}$$

where U is an open set of $\mathfrak{g}^*$, $2r + h = m$, φ a Weinstein chart. Denote $x_k = f(X_k)$, $k \in \{1, \dots, n\}$ for some f of $\mathfrak{g}^*$, we have

$$\begin{aligned} p_i &= p_i(x_{i_1}, \dots, x_{i_h}, x_{j_1}, \dots, x_{j_d}) \in C^\infty(U), \\ \tilde{p}_i &= \tilde{p}_i(x_{i_1}, \dots, x_{i_h}) \end{aligned}$$

considered only as function of (x_{i_h}). We proceed to a Taylor expansion $\tilde{p}_i$ at the point

$$R = (R_{i_l}(\mu, x_{j_1}, \dots, x_{j_d}))_{l \in \{1, \dots, h\}}.$$

We have

$$\tilde{p}_i(\tilde{x}) = \tilde{p}_i(R) + \sum\nolimits_{|\alpha| \le 1} \frac{\partial^\alpha \tilde{p}_i}{\partial \tilde{x}^\alpha}(R)(\tilde{x} - R)^\alpha,$$

where

$$\begin{aligned} &\tilde{x} = x_{i_1}, \dots, x_{i_h}, \quad \alpha = (\alpha_1, \dots, \alpha_h) \in \mathbb{N}^h, \quad \tilde{x}^\alpha = x_{i_1}^{\alpha_1}, \dots, x_{i_h}^{\alpha_h}, \\ &\alpha! = \alpha_1! \cdots \alpha_h!, \quad |\alpha| = \alpha_1 + \cdots + \alpha_h. \end{aligned}$$

Next, $\partial^\alpha P_i/\partial\tilde{x}^\alpha(R)$ is a a function of $(x_{j_1},\ldots,x_{j_d})$. We cannot assert that the $\partial^\alpha P_i/\partial\tilde{x}^\alpha(R)$ are polynomial in the variables $x_{j_1},\ldots,x_{j_d}$. Then we develop $\partial^\alpha P_i/\partial\tilde{x}^\alpha(R)$ at $(\mu_{j_1},\ldots,\mu_{j_d})$. Finally, we obtain an association to p_i an element denoted $\widehat{p_i}$ by the substitution of the indeterminates X_i for the coordinates x_i. We do not know if this element $\widehat{p_i}$ is in $\widehat{S}=\mathbb{R}[T_{j_1},\ldots,T_{j_d}][[U_{i_1},\ldots,U_{j_h}]]$, but we have

$$\widehat{p_i}\in\mathbb{R}[[T_{j_1},\ldots,T_{j_d}]][[U_{i_1},\ldots,U_{j_h}]]=\mathbb{R}[[T,U]].$$

Similarly, we obtain an association to q_i and z_l the elements $\widehat{q_i}$ and $\widehat{z_l}$ of $\mathbb{R}[[T,U]]$, for $i,j\in\{1,\ldots,n\}$, $l\in\{1,\ldots,h\}$, which satisfy the relations $(*)$ of Weinstein's splitting theorem. The algebra $\mathbb{R}[[\widehat{z_1},\ldots,\widehat{z_h}]]$ is the algebra obtained by expansion of Weinstein's commutant in formal power series, we denote it by C_{W}.

Thus the commutants C and C_{W} are included in the same algebra formal power series $\mathbb{R}[[T,U]]$, which permits us to compare them.

Theorem 44. *Let $\mathfrak{g}$ be a finite-dimensional $\mathbb{R}$-nilpotent Lie algebra. Let $\mu\in\mathfrak{g}^*$ and $I(\mu)$ be the rational Poisson ideal associated to μ. Let $\varphi: S(\mathfrak{g})/I(\mu)\longrightarrow\widehat{S}=\varprojlim_n S(\mathfrak{g})/I^n(\mu)$ be a lifting homomorphism of Poisson algebras. Let (p_i,q_j,z_α) be a Weinstein chart of Poisson algebras at μ and let C_{W} the formal transverse algebra to the leaf through μ obtained by expansion in formal series of the Weinstein commutant. Then we have a Poisson algebras isomorphism between the algebra C_{W} and the commutant $\varphi(S(\mathfrak{g})/I(\mu))$ in $\widehat{S}$.*

$$C_{\mathrm{W}}\simeq C(\varphi\big(S(\mathfrak{g})/I(\mu)),\widehat{S}).$$

Proof. Let $\mathfrak{g}=\sum_{i=1}^{i=m}\mathbb{R}X_i$. The element μ of $\mathfrak{g}^*$ belongs to some stratum Ω_e for some $e=\{j_1<\cdots<j_d\}$, d being the dimension of the orbit through μ. Let $\overline{e}=\{1,\ldots,m\}-e=\{i_1<\cdots<i_h\}$. We know that the associated ideal to μ is

$$I(\mu)=\sum_{l=1}^{l=h}l(X_{i_l}-R^e_{i_l}(\mu,X_{j_1},X_{j_2},\ldots,X_{j_\alpha}))S(\mathfrak{g}),$$

with $j_\alpha\le i_l<j_{\alpha+1}$. Let us say $T_{j_k}=X_{j_k}-\mu_{j_k}$, $k\in\{1,\ldots,k\}$ and $U_{i_l}=X_{i_l}-R^e_{i_l}(\mu,X_{j_1},X_{j_2},\ldots,X_{j_\alpha})$, $l\in\{1,\ldots,h\}$. The choice of a lifting homomorphism φ provide elements $\widehat{S}$, (a_i,b_j,c_α), $i,j\in 1,\ldots,d,\alpha\in 1,\ldots,m-2r$, satisfying Weinstein's relations. We have

$$\widehat{S}=\mathbb{R}[T_{j_1},\ldots,T_{j_d}][[U_{i_1},\ldots,U_{i_h}]]\subset\mathbb{R}[[T,U]],$$

the elements a_i,b_j,c_α which form a Weinstein's formal system of the algebra of formal series $\mathbb{R}[[T,U]]$ (see Theorem 43 (Weinstein's formal theorem)). We have

$$\mathbb{R}[[a,b,c]]\simeq\mathbb{R}[[T,U]]\quad\text{and}\quad C=\mathbb{R}[[c]].$$

Let (p_i,q_j,z_α) be a Weinstein chart at μ and the elements $(\widehat{p_i},\widehat{q_j},\widehat{z_\alpha})$ of $\mathbb{R}[[T,U]]$ associated by expansion in Taylor series as above. These elements form a Weinstein formal system of the algebra $\mathbb{R}[[T,U]]$. We have

$$\mathbb{R}[[\widehat{p},\widehat{q},\widehat{z}]]\simeq\mathbb{R}[[T,U]]\quad\text{et}\quad C_{\mathrm{W}}=\mathbb{R}[[\widehat{z}]].$$

By uniqueness of the formal transverse Poisson algebra in Weinstein's formal theorem (Theorem 43), the formal transverse Poisson algebras $\mathbb{R}[[c]]$ and $\mathbb{R}[[\widehat{z}]]$ are isomorphic. Thus the commutant C and the formal transverse algebra to the leaf through μ, C_W, are isomorphic. □

We have the immediate following consequence.

Corollary 45. *With the hypothesis of Theorem 44, let $\mu \in g^*$ and $I(\mu)$ be the Poisson ideal associated to μ. Let $\varphi : S(\mathfrak{g})/I(\mu) \longrightarrow \widehat{S}$ be a lifting theorem (Theorem 17). Let C be the commutant of $\varphi(S(\mathfrak{g})/I(\mu))$ in $\widehat{S}$.*

If μ belongs to an orbit of maximal dimension, the corresponding commutant C is a trivial Poisson algebra of formal series with dimension the codimension of the orbit.

Example 46 (cf. [31]). We will find the commutant C from algebraic methods in Section 3.1.5. Weinstein's theorem allows us to find the commutant generators directly.

3.2.3. Application of Dirac's formula to $\mathfrak{g}^*$

Using Dirac's bracket formula of Theorem 40, We are going to clarify the transverse Poisson structure. With the aid of the isomorphism given in Theorem 44, this formula calculates with ease the commutant C associated to a lifting homomorphism when we compare to the algebraic method of successive lifting homomorphisms. We will even have a computation algorithm. Let $\mathfrak{g}$ be a Lie $\mathbb{R}$-algebra, hence we apply Dirac's formula to the manifold $\mathfrak{g}^*$, endowed with its Lie–Poisson structure which the Poisson tensor field Λ is such that

$$\Lambda_\mu^\# : \mathfrak{g} \longrightarrow \mathfrak{g}^*, \quad \mu \in \mathfrak{g}^* \quad \text{et} \quad \Lambda^\sharp(x) = \mu([x, \cdot]).$$

The characteristic space at μ, Im $\Lambda_\mu^{\sharp}$ is $\mathfrak{g}_\mu^\perp$, $\mathfrak{g}_\mu^\perp$ being the annihilator of the stabilizer $\mathfrak{g}_\mu$ at μ in $\mathfrak{g}^*$. Thus, the characteristic space to the symplectic leaf through μ, this one being denoted Ω_μ, is $\mathfrak{g}_\mu^\perp$. Let m be a complement of $\mathfrak{g}_\mu$ in $\mathfrak{g}$. From the decomposition $\mathfrak{g} = \mathfrak{g}_\mu \oplus m$, we have $\mathfrak{g}^* = \mathfrak{g}_\mu^\perp \oplus m^\perp$, id est

$$T_\mu \mathfrak{g}^* = T_\mu \Omega_\mu \oplus T_\mu(\mu + m^\perp).$$

Therefore the manifolds $\mu + m^\perp$ and Ω_μ are transverse submanifolds at μ.

Theorem 47. *Let $\mathfrak{g}$ be a finite-dimensional Lie $\mathbb{R}$-algebra. Let Ω a symplectic in $\mathfrak{g}^*$. Let $\mu \in \Omega$, m be a complement of $\mathfrak{g}_\mu$ in $\mathfrak{g}$, $(Z_1, \ldots, Z_k)$ a basis of g_μ and $(X_{k+1}, \ldots, X_n)$ a basis of m. Let $N = \mu + m^\perp$, be the transverse submanifold at μ to the leaf and endowed with the Poisson structure induced by that of $\mathfrak{g}^*$. We choose $(z_1, \ldots, z_k)$ as a local coordinate system of N with $z_i = \varphi(Z_i)$, $\varphi \in \mu + m^\perp$. Then, expressed in the local coordinate system chosen for N, the component of the tensor Λ_N of the induced Poisson structure are rational fractions*

$$\Lambda_{N_{(ij)}} = \{z_i, z_j\}_N \in \mathbb{R}(z_1, \ldots, z_k), \quad i, j \in \{1, \ldots, k\}.$$

Proof. Let $\sigma : \mathfrak{g}^* \longrightarrow \mathbb{R}^n$ be the system of linear coordinates such that for φ in $\mathfrak{g}^*$:

$$\sigma(\varphi) = (\varphi(Z_1), \dots, \varphi(Z_k), \varphi(X_{k+1}), \dots, \varphi(X_n)) = (z_1, \dots, z_k, x_{k+1}, \dots, x_n).$$

We notice that the choice of the basis to express the coordinates is dictated by the fact that for all φ of the submanifold $\mu + m^{\perp}$ transverse to the leaf through μ, we have

$$\varphi(X_i) = \mu_i, \quad i \in \{k+1, n\}.$$

Let U be an open set in $\mathfrak{g}^*$ and let $N = \mu + m^{\perp}$ be the transverse submanifold to the leaf at $\mu = (\mu_1, \dots, \mu_n)$, we have

$$N \cap U = \{\varphi \in U, \quad x_{k+1} = \mu_{k+1}, \dots, x_n = \mu_n\}.$$

The map $\widetilde{\sigma} : N \cap U \longrightarrow \mathbb{R}^k$ such that

$$\widetilde{\sigma}(\varphi)(\varphi(Z_1), \dots, \varphi(Z_k)) = (z_1, \dots, z_k)$$

is a chart of N. From Dirac's formula, we have for $\varphi \in U \cap N$, $i, j \in \{1, \dots, k\}$, $\alpha, \beta \in \{k+1, \dots, n\}$ using Einstein's convention $\{z_i, z_j\}_{U \cap N}(\varphi) = \{z_i, z_j\}_{\mathfrak{g}^*}(\varphi) - \{z_i, x_\alpha\}_{\mathfrak{g}^*}(\varphi) C^{\alpha\beta}(\varphi)\{x_\beta, z_j\}(\varphi)$, where if $C(\varphi)$ is the matrix $[\{x_\alpha, x_\beta\}_{\mathfrak{g}^*}(\varphi)]$, then $C^{-1}(\varphi) = [C^{\alpha\beta}(\varphi)]$ is the inverse matrix. We have $\{z_i, z_j\}_{\mathfrak{g}^*}(\varphi) = \varphi([Z_i, Z_j]) = a^l_{ij}\varphi(Z_l)$ where the $(a^l_{i,j})$ are the constants of structure of $\mathfrak{g}$, and l, i, j belong to $\{1, k\}$. We see that $\{z_i, z_j\}_{\mathfrak{g}^*}(\varphi) = a^l_{i,j} z_l$ is a linear expression of z. Similarly $\{z_i, x_\alpha\}(\varphi) = a^l_{i\alpha} z_l + a^{\gamma}_{i\alpha}\mu_\gamma$ is an affine expression of z. It is remaining to verify $C^{\alpha\beta}(\varphi)$. We have the equalities $C_{\alpha\beta}(\varphi) = \{x_\alpha, x_\beta\}(\varphi) = \varphi([X_\alpha, X_\beta]) = a^l_{\alpha\beta} z_l + a^{\gamma}_{\alpha\beta}\mu_\gamma$. The matrix $C^{-1}(\varphi) = [C^{\alpha\beta}(\varphi)]$ being the inverse matrix $C(\varphi) = [C_{\alpha\beta}(\varphi)]$, the coefficients $C^{\alpha\beta}(\varphi)$ are rational fractions in $z_1, \dots, z_k$. It is hence clear that $\{z^i, z^j\}$ belongs to $\mathbb{R}(z_1, \dots, z_h)$. □

Practical calculus. For φ in $N = \mu + m^{\perp}$, we define the matrix

$$\begin{array}{c} \\ g_\mu \updownarrow \\ m \updownarrow \end{array} \begin{pmatrix} \overset{g_\mu}{\longleftrightarrow} & \overset{m}{\longleftrightarrow} \\ \{z_i, z_j\}(\varphi) & \{z_i, x_\alpha\}(\varphi) \\ \{x_\alpha, z_j\}(\varphi) & \{x_\alpha, x_\beta\}(\varphi) \end{pmatrix} = \begin{bmatrix} A(\varphi) & B(\varphi) \\ D(\varphi) & C(\varphi) \end{bmatrix}$$

such that $D = -{}^tB$.

Expressed in the coordinates (z_i), the matrix of the tensor Λ_N of the Poisson structure induced on N by $\mathfrak{g}^*$ is

$$\Lambda_N(\varphi) = A(\varphi) - B(\varphi)C^{-1}(\varphi)D(\varphi).$$

This formula provides an algorithm calculating the transverse structure, using Pedersen's computations. It is sufficient to calculate the stabilizer.

Example 48. Let $\mathfrak{g}_{53} = \sum_{i=1}^{i=5}\mathbb{R}X_i$ (see [26, p. 20]) be the Lie algebra with non-null brackets $[X_i, X_j]$, $i > j$, $[X_5, X_4] = X_2$, $[X_5, X_2] = X_1$, $[X_4, X_3] = X_1$. We calculate the transverse structure at a point μ of the stratum Ω_e defined by $\Omega_e = \{\sum_{i=1}^{i=5}\xi_i X^{i*}\colon \xi_1 =$

$0, \xi_2 \neq 0\}$, with $e = \{j_1 < j_2\} = \{4, 5\}$. The stabilizer of μ being $\mathfrak{g}_\mu = \mathbb{R}X_1 \oplus \mathbb{R}X_2 \oplus \mathbb{R}X_3$, we choose a complement m of $\mathfrak{g}_\mu$ in $\mathfrak{g}_{53}$, $m = \mathbb{R}X_4 \oplus \mathbb{R}X_5$. We choose the linear coordinates $\sigma : \mathfrak{g}^*{}_{53} \longrightarrow \mathbb{R}^5$ such that for φ in $\mathfrak{g}^*{}_{53}$ we have $\sigma(\varphi) = (\varphi(X_1), \dots, \varphi(X_5)) = (z_1, z_2, z_3, x_4, x_5)$. Let N be the submanifold transverse through μ and U open set including μ, we have $N \cap U = \{\varphi \in U;\ x_4 = \mu_4, x_5 = \mu_5\}$ with $\mu = \sum_{i=1}^{i=5} \mu_i X^{i*}$. An element of $\mu + m^\perp$ has for coordinates $(z_1, z_2 + \mu_2, z_3 + \mu_3, \mu_4, \mu_5)$. The tensor Λ_N induced on N by $\mathfrak{g}^*{}_{53}$, expressed in the linear coordinates, is $\Lambda_N(\varphi) = A(\varphi) + B(\varphi) C^{-1\,t} B(\varphi)$ where $A(\varphi)$ is the matrix $[\{z_i, z_j\}(\varphi)]$, $i, j \in \{1, 2, 3\}$, $B(\varphi)$ is the matrix $[\{z_i, x_\alpha\}(\varphi)]$, $i \in \{1, 2, 3\}$ et $\alpha \in \{4, 5\}$ and where the matrix $C^{-1}(\varphi)$ is such that $C(\varphi) = [\{x_\alpha, x_\beta\}(\varphi)]$, $\alpha, \beta \in \{4, 5\}$. We find

$$\Lambda_N(z_1, z_2, z_3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -z_1^2/(z_2 + \mu_2) \\ 0 & z_1^2/(z_2 + \mu_2) & 0 \end{bmatrix}.$$

The structure is defined by the bracket $\{z_3, z_2\}_N = z_1^2/(z_2 + \mu_2) \in \mathbb{R}(z_1, z_2, z_3)$.

3.2.4. Comparison of the commutant m-adic graded algebra with the symmetric algebra $S(\mathfrak{g}_\mu)$ of the stabilizer at an orbit point

Theorem 49. *Let $\mathfrak{g}$ be a nilpotent Lie $\mathbb{R}$-algebra. Let $\mu \in \mathfrak{g}^*$ and $I(\mu)$ the Poisson ideal associated to μ. Let $\varphi : S(\mathfrak{g})/I(\mu) \longrightarrow \widehat{S}$ be a lifting homomorphism (Theorem 17). Let C be the commutant of $\varphi(S(\mathfrak{g})/I(\mu))$ in $\widehat{S}$ and $\widehat{I} \cap C$ its maximal ideal denoted by m. Then*
1. *the symmetric Poisson algebra of $\mathfrak{g}_\mu$ is isomorphic to the m-adic graded algebra of C: $\mathrm{Gr}_m C \simeq S(\mathfrak{g}_\mu)$;*
2. *we have the isomorphism of Poisson algebras $\mathrm{Gr}_{I(\mu)} S(\mathfrak{g}) \simeq S(\mathfrak{g})/I(\mu) \otimes S(\mathfrak{g}_\mu)$.*

Proof. (1) According to Theorem 44 we can assert that the commutant C is isomorphic to the formal Poisson algebra C_{W} transverse to the leaf through μ get by the formal power series expansion of Weinstein commutant. We can calculate C_{W} with the aid of Dirac's formula. Using this last ease (see Theorem 47), we choose a complement h de $\mathfrak{g}_\mu$ in $\mathfrak{g}^*$, a basis $(Z_1, \dots, Z_k)$ of $\mathfrak{g}_\mu$, a basis $(X_{k+1}, \dots, X_n)$ of h and $(z_1, \dots, z_k, x_{k+1}, \dots, x_n)$ the associated linear coordinates. We have, for $\varphi \in \mu + h^\perp$, $i, j \in \{1, \dots, k\}$, $\alpha, \beta \in \{k+1, \dots, n\}$ using Einstein's convention

$$\{z_i, z_j\}_{\mu + h^\perp}(\varphi) = \{z_i, z_j\}_{\mathfrak{g}^*}(\varphi) - \{z_i, x_\alpha\}_{\mathfrak{g}^*}(\varphi) C^{\alpha\beta}(\varphi) \{x_\beta, z_j\}(\varphi),$$

where if $C(\varphi)$ is the matrix $[\{x_\alpha, x_\beta\}_{\mathfrak{g}^*}(\varphi)]$, then $C^{-1}(\varphi) = [C^{\alpha\beta}(\varphi)]$ is the inverse matrix. We know that the $\{z_i, z_j\}_{\mu + h^\perp}$ are rational fractions, thus we shall have

$$C_{\mathrm{W}} = k[[z_1, \dots, z_k]].$$

Poisson algebra with maximal ideal $m_{\mathrm{W}} = (z_1, \dots, z_k)$ and with the brackets $\{z_i, z_j\}_{C_{\mathrm{W}}}$ given by Taylor's expansion of the above formula. If we have the inclusion $[\mathfrak{g}_\mu, h] \subset h$, then

$$\{z_i, z_j\}_{C_{\mathrm{W}}} = \{z_i, z_j\}_{\mathfrak{g}^*{}_\mu}$$

or else

$$\{z_i, z_j\}_{C_W} = \{z_i, z_j\}_{\mathfrak{g}^*_\mu} + S_{ij},$$

where S_{ij} is a formal power series in the z_i with valuation $v(S_{ij})$ such that $v(S_{ij}) \geq 2$. The m-adic graded algebra $\mathrm{Gr}_{m_W} C_W$ is generated as $\mathrm{Gr}_0(C_W)$-algebra by $\mathrm{Gr}_1(C_W) = m_W/m_W^2$. We have $\mathrm{Gr}_{m_W} C_W = k[z_1 + m_W^2, z_2 + m_W^2, \ldots, z_k + m_W^2]$ with $\{z_i + m_W^2, z_j + m_W^2\} = \{z_i, z_j\}_{\mathfrak{g}^*_\mu} + m_W^2$. Thus we obtain $\mathrm{Gr}_{m_W} C_W \simeq S(\mathfrak{g}_\mu)$. From the isomorphism $\mathrm{Gr}_{m_W} C_W \simeq \mathrm{Gr}_m C$, we obtain $\mathrm{Gr}_m C \simeq S(\mathfrak{g}_\mu)$. (2) Follows from Theorem 27. □

Example 50. Let $\mathfrak{g}^*_{67} = \sum_{i=1}^{i=6} \mathbb{R} X_i$ be the nilpotent Lie algebra (see [26, p. 59]) with non-null brackets $[X_i, X_j]$, $i > j$, $[X_6, X_5] = X_3$, $[X_6, X_3] = X_2$, $[X_6, X_3] = X_1$, $[X_5, X_4] = X_2$, $[X_4, X_3] = -X_1$. Let $e = \{5, 6\}$ and let μ be the point of the stratum Ω_e defined by $\Omega_e = \{\sum_{i=1}^{i=6} \xi_i X^{i*}; \xi_1 = 0, \xi_2 = 0, \xi_3 \neq 0\}$. The stabilizer of μ being $\mathfrak{g}_\mu = \mathbb{R}X_1 \oplus \mathbb{R}X_2 \oplus \mathbb{R}X_3 \oplus \mathbb{R}X_4$, we choose m a complement of $\mathfrak{g}_\mu$ in $\mathfrak{g}_{67}$, $m = \mathbb{R}X_5 \oplus \mathbb{R}X_6$. We choose the linear coordinates $\sigma : \mathfrak{g}^*_{67} \longrightarrow \mathbb{R}^5$ such that for φ in $\mathfrak{g}^*_{67}$ we have $\sigma(\varphi) = (\varphi(X_1), \ldots, \varphi(X_6)) = (z_1, z_2, z_3, z_4, x_5, x_6)$. An element of $\mu + m^\perp$ has for coordinates $(z_1, z_2, z_3 + \mu_3, z_4 + \mu_4, \mu_5, \mu_6)$. By virtue of Dirac's brackets formula, we obtain, expressed in the linear coordinates, the tensor Λ_N induces on N (see Example 48). Thus the transverse structure is defined by $\{z_4, z_2\}_N = (z_1 z_2/z_3 + \mu_3)$ et $\{z_4, z_3\}_N = -z_1 + (z_2^2/z_3 + \mu_3)$. The m-adic graded algebra of the commutant C is the polynomial algebra $k[z_1, z_2, z_3, z_4]$, endowed with the Poisson structure defined by the non-null brackets $\{z_3, z_4\} = z_1$. The algebra $S(\mathfrak{g}_\mu)$ is the polynomial algebra $k[X_1, X_2, X_3, X_4]$ endowed with the Poisson structure defined by the non-null brackets $\{X_3, X_4\} = X_1$. We verify that

$$\mathrm{Gr}_m C \simeq S(\mathfrak{g}_\mu).$$

3.2.5. Quantization example

Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra, $U(\mathfrak{g})$ its enveloping algebra and J a rational ideal on k. The algebra $U(\mathfrak{g})/J$ an Weyl algebra. In Section II of [11], Fokko du Cloux has shown that there exists a homomorphism of unitary associative algebras $\phi : U(\mathfrak{g})/J \longrightarrow \varprojlim_n U(\mathfrak{g})/J^n = \widehat{U(\mathfrak{g})}$ such that $f_1 \circ \phi = \mathrm{Id}_{U/J}$, where $f_1 : \widehat{U} \longrightarrow U/J$ is the canonical projection. We shall denote by D the commutant associated to this lifting homomorphism,

$$D = \{u \in \widehat{U} : [\phi(a), u] = 0 \quad \forall a \in U/J\},$$

where the bracket is defined $[a, b] = ab - ba$ for a, b in $\widehat{U}$.

Fokko du Cloux has dealt in detail the case of the Lie algebra $\mathfrak{g}_{53}$ [11, pp. 196–199]. We have seen in the above example that the lifting homomorphim formulae from $U(\mathfrak{g}_{53})/J$ into $\widehat{U}$ are identical to the lifting formulae from $S(\mathfrak{g}_{53})/I$ into $\widehat{S}$, the ideals I and J being in correspondence with the help of Dixmier's map [8, 6.3.3, p. 195]. For the commutants D in $\widehat{U}$ and C in $\widehat{S}$, the formulae gave the generators of the respective maximal ideals are also stricly the same.

Let us consider this similarity in more details. In the algebra $\widehat{U}$, for $J = (X_1, X_2 - \mu_2, X_3 - \mu_3)$, the commutant is the non-commutative formal power series algebra $k[[z_1, z_2, z_3]]$ such that z_1 is central and $[z_3, z_2] = z_1^2(z_2 + \mu_2)^{-1}$.

In the algebra $\widehat{S}$, for $I = (X_1, X_2 - \mu_2, X_3 - \mu_3)$, the commutant is the formal power series algebra $k[[z_1, z_2, z_3]]$ such that z_1 is central and $\{z_3, z_2\} = z_1^2(z_2 + \mu_2)^{-1}$. We notice that in $\widehat{U}$ we have $[z_3(z_2 + \mu_2), z_2] = z_1^2$. If we set Set $z_3' = z_3(z_2 + \mu_2)$, we get the isomorphism $k[[z_1, z_2, z_3]] \simeq k[[z_1, z_2, z_3']]$ with

$$[z_3', z_2] = z_1^2.$$

The same holds in $\widehat{S}$, we have $\{z_3(z_2 + \mu_2), z_2\} = z_1^2$. Settling $z_3' = z_3(z_2 + \mu_2)$, we get the isomorphism $k[[z_1, z_2, z_3]] \simeq k[[z_1, z_2, z_3']]$ with

$$\{z_3', z_2\} = z_1^2.$$

Let $\mathfrak{A}_t$, $k[[z_1, z_2, z_3']]$ be the algebras such that $z_1$is central and $[z_3', z_2] = tz_1^2$. Then $\mathfrak{A}_t$ is a deformation (see [10] for a precise definition) of $\mathfrak{A}_0 = k[[z_1, z_2, z_3']]$, Poisson algebra endowed with the bracket

$$\{z_3', z_2\} = z_1^2.$$

In this example, the associative algebra D is a quantization of the Poisson algebra C. In the general case we can conjecture: "*Fokko du Cloux's is a quantization of the transverse structure*". The comparison of the commutant D and C is not easy. The symmetrization $\omega : S \longrightarrow U$ does not map the ideal I on the ideal J although we can find generators of I such their images by ω generate J (voir [14]).

Noticing the similarity between the formulas which give the commutants, we are tempted to use Dirac's formula to the search of the commutant D in $\widehat{U}$. The obstruction is to define the inverse matrix which appears in Dirac's formula.

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